KEY

TO THE

ELEMENTS OF EUCLID

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BY

JOHN STURGEON MACKAY, M.A., F.R.S.E.

MATHEMATICAL MASTER IN THE EDINBURGH ACADEMY

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PREFACE.

In consequence of the large number of deductions, it was impossible to furnish diagrams without materially increasing the size and price of the present volume; but it is hoped that the descriptions given of the figures that require to be made are sufficiently full and exact to cause no difficulty in understanding the solution. De Morgan says: 'I am satisfied, from sufficient trial, that when proper description of the diagram is given in the text, the person who draws his own diagram from the text, will arrive at the author's meaning in half the time which is employed by another, to whom the successive appearance of the parts is prevented by his seeing the whole from the beginning.' It is perhaps hardly necessary to add that in the solutions of the 'riders,' the figures of the *Euclid* have been utilised as far as possible.

Edinburgh Academy,
July 1885.

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ANSWERS TO THE QUESTIONS

ON THE

DEFINITIONS, POSTULATES, AXIOMS.

1. By a dot with a letter attached. 2. Position. Magnitude.
3. No; because geometrical points have no size.
4. See the first figure to Definition 5. 5. No.
6. A straight line is that which lies evenly to the points within itself.
7. No; because geometrical lines have no breadth.
8. On the surface. 9. On the surface.
10. Yes; on the surface of a cylindrical ruler.
11. Two. 12. The vertex. 13. No.
14. In two ways. The letter at the vertex must be in the middle.
15. When several angles have the same vertex.
16. By the letter at the vertex.
17. *AOB*, *BOC*, *AOC*. 18. *AOB*, *BOC*, *COA*.
19. *AOB*, *AOC*, *AOD*, *BOC*, *BOD*, *COD*.
20. *AOB*, *BOC*, *COD*, *AOC*, *BOD*, *AOD*.
21. *DAE*, *DAG*, *DAC* (or *EAD*, *GAD*, *CAD*), *FAE*, *FAG*, *FAC* (or &c.), *BAE*, *BAG*, *BAC* (or &c.), and *A*.
22. None. 23. *AOC*. 24. *BOC*. 25. *AOB*.
26. (*a*) *AOC*; *AOD*; *AOD*; *BOD*. (*b*) *AOC*; *BOD*; *COD*; *AOB*. (*c*) *BOC*; *COD*. (*d*) *AOD*; *AOB*.
27. See angles *A* and *B* at the foot of p. 3, *Elements of Euclid*.

28. In the figure to Question 17, if OB is equal to OC, angles AOB and AOC are unequal angles with equal arms.
29. No. See figure to Definition 8.
30. The bisector of the sum of the two angles.
31. An obtuse angle. 32. An acute angle. 33. A right angle.
34. ABC, ABD; ABE, EBD; EBC.
35. Adjacent; vertically opposite; adjacent; vertically opposite; adjacent.
36. No. 37. See *Elements of Euclid*, p. 68.
38. See *Elements of Euclid*, p. 64.
39. See *Elements of Euclid*, p. 47 and p. 41.
40. One. A circle. 41. Two.
42. By the definition of a circle.
43. All radii of a circle are equal, by definition;
    all diameters of a circle are double of a radius;
    therefore all diameters of a circle are equal.
44. Not necessarily.
45. A circle is an inclosed space, and the circumference is the line which incloses it.
46. Sometimes. 47. Three.
48. No; because the two radii might not be in the same straight line.
49. Let ABC be a circle whose centre is O (fig. 1, p. 154),
    and let P be a point inside the circle.
    Through O and P draw the straight line OP,
    and let it meet the circumference at A.
    Then OP is less than OA.
    But OA is a radius of the circle ABC;
    therefore OP is less than a radius.
50. Let ABC be a circle whose centre is O (fig. 2, p. 154),
    and let P be a point outside the circle.
    Through O and P draw the straight line OP,
    and let it meet the circumference at A.
    Then OP is greater than OA.
    But OA is a radius of the circle ABC;
    therefore OP is greater than a radius.
51. Three. 52. Rectilineal figures. 53. Yes. 54. Three.
55. BCA, CAB, ACB, BAC, CBA. 56. An isosceles triangle.
57. An equilateral triangle. 58. A scalene triangle.
59. The perimeter. 60. Any side may be so called.
61. That which is neither of the equal sides.

62. A right-angled triangle.
63. The base, and the perpendicular. 64. Yes. 65. *C, A, B.*
66. *ACB, BAC, ABC.* 67. *BC, CA, AB.*
68. *A, BAC; B, ABC; C, ACB.*
69. *ABC, DBC, ABE, DCE, EBC.* 70. *ABD, ACD, EAD.*
71. *BAC, BDC, BEC; BAE, BCE; CDE, CBE.*
72. *BE, BC; CE, CB.*
73. *AEB, DEC; AED, BEC; AED, BEC.*
74. *BCDA, CDAB, DABC, ADCB, DCBA, CBAD, BADC.*
75. Eight triangles. *AEB, BEC, CED, DEA, ABC, ADC, BAD, BCD.*
76. *ABC, ADC.* 77. *BAD, BCD.*
78. *BAD, BCD.* 79. *ABC, ADC.*
80. Yes; because it is a rhombus and something more.
81. No; because it may not have its angles right.
82. Yes; because it is a parallelogram and something more.
83. No; because it may not have its angles right.
84. No; because other figures than quadrilaterals may have their opposite sides parallel.
85. Yes; because each is a trapezium and something more.
86. No; because it may have only one pair of sides parallel.
87. A straight line joining two opposite corners. Two.
88. Any number more than four; but sometimes figures of three and four sides are called polygons.
89. The first. 90. The second. 91. The third.
92. In the sense of circumference.
93. The ruler and the compasses. They are not to be used to carry distances.
94. A self-evident truth. Things which are equal to the same thing are equal to one another.
95. Magnitudes which coincide with one another are equal to one another.
96. Scarcely; because straight lines and angles cannot be said to fill space.
97. All right angles are equal to one another.
98. Two straight lines which intersect one another cannot be both parallel to the same straight line.
99. Yes. 100. Yes.

DEDUCTIONS.

BOOK I.

PROPOSITION 1.

1. Because $AB = AF$, being radii of circle BCD, *I. Def.* 16
   and $AB = BF$, being radii of circle ACE; *I. Def.* 16
   $\therefore$ $AF = BF$, *I. Ax.* 1
   $\therefore$ $AB = AF = BF$,
   and ABF is an equilateral triangle. *I. Def.* 23
2. Let A and B be the two points.
   Join AB, and describe on it an equilateral $\triangle$ ABC. *I.* 1
   C is the point required.
3. Let AB be the given straight line.
   Describe on AB two equilateral $\triangle$s ABC, ABF. *I.* 1
   $ACBF$ is the required rhombus.
4. Let AB be the given straight line.
   Describe on AB an equilateral $\triangle$ ABC, *I.* 1
   and on BC describe an equilateral $\triangle$ BCE. *I.* 1
   $ABEC$ is the required rhombus.
5. Because $AD = AC$, being radii of circle BCD, *I. Def.* 16
   and $BE = BC$, being radii of circle ACE; *I. Def.* 16
   $\therefore AD + AB + BE = AC + AB + BC$,
   that is, $DE = AC + AB + BC$.
6. Let ABC be the given triangle.
   With centre B and radius BA, describe circle ADF.
   With centre C and radius CA, describe circle AEG;
   produce BC both ways to meet the circles at D and E.
   DE is the straight line required.
   Because $AB = BD$, being radii of circle ADF, *I. Def.* 16
   and $CA = CE$, being radii of circle AEG; *I. Def.* 16
   $\therefore AB + BC + CA = BD + BC + CE$,
   $= DE$.

7. Let AB be the given straight line.
With A as centre, and AB as radius, describe circle BCD; produce BA to meet the circle at D.
BD is the straight line required.
Because $AD = AB$, being radii of circle BCD, *I. Def.* 16
$\therefore BD = AB + AB = 2\,AB$.

8. Let AB be the given straight line.
Make the same construction as in I. 1,
and produce AB both ways to meet the circles at D and E.
DE is the straight line required.
Because $AD = AB$, being radii of circle BCD, *I. Def.* 16
and $BE = AB$, being radii of circle ACE; *I. Def.* 16
$\therefore DE = AB + AB + AB = 3\,AB$.

9. Let AB be the given straight line.
Find by the seventh deduction $BD = 2\,AB$;
then find another straight line $= 2\,BD$,
and therefore $= 4\,AB$.

10. Let AB be the given straight line.
Find $DE = 3\,AB$ by the eighth deduction.
With D as centre, and DA as radius, describe circle AGH.
With E as centre, and EB as radius, describe circle BKL; produce DE both ways to meet these circles at H and L.
HL is the straight line required.
Because $HD = DA = AB$,
and $EL = EB = AB$;
$\therefore HL = HD + DA + AB + EB + EL = 5\,AB$.

PROPOSITION 2.

1. Since $DE =$ twice BC (*Hyp.*), and BC is a radius of the small circle; $\therefore DE$ must be a diameter of the small circle;
$\therefore D$ must be situated on the $\bigcirc^{ce}$ of the small circle,
and BD must be a radius of the small circle.
Now, since $BA = BD$, BA must be a radius of the small circle;
$\therefore$ the given point A must be on the $\bigcirc^{ce}$ of the small circle.

2. With A as centre, and AB as radius, describe a circle.
Join A to any number of points on the $\bigcirc^{ce}$ of this circle.
These straight lines will all be $= AB$, for they are radii of the same circle. *I. Def.* 16

3. With B as centre, and BA as radius, describe a circle.
Join B to any number of points on the ○ᶜᵉ of this circle.

4. With A as centre, and AB as radius, describe a circle.
Through A draw any number of diameters of the circle.
These diameters will all be double of AB, for AB is a radius.

5. With B as centre, and BA as radius, describe a circle.
Through B draw any number of diameters of the circle.

6. Let AB be the given base, C the given straight line.
From A draw $AD = C$, and from B draw $BE = C$, *I.* 2
With A as centre, and AD as radius, describe a circle;
with B as centre, and BE as radius, describe a circle;
cutting the former circle at F. Join FA, FB.
Then $AF = AD$ (*I. Def.* 16) $= C$. *Const.*
$BF = BE$ (*I. Def.* 16) $= C$. *Const.*
$\therefore$ $AF = BF$, and $\triangle\ ABF$ is isosceles.
The length of C must be greater than half of AB, otherwise the two circles will not cut each other.

7. In connection with fig. 1, repeat, word for word, the construction and proof given in the text.

8, 9. In connection with figs. 2 and 3, repeat, word for word, the construction and proof given in the text, substituting, however, the whole $BE =$ the whole AG *I. Ax.* 2
for remainder $BE =$ remainder AG. *I. Ax.* 3

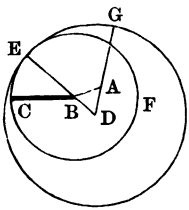


Fig. 1.

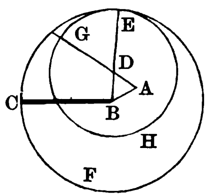


Fig. 2.

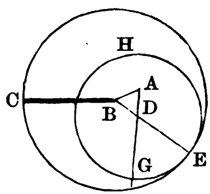


Fig. 3.

10. In connection with figs. 4 and 5, write out, word for word, the construction and proof given in the text; but wherever B occurs substitute C, and wherever C occurs substitute B.
In connection with figs. 6 and 7 do the same thing, but substitute also the whole $CE =$ the whole AG *I. Ax.* 2
for remainder $CE =$ remainder AG. *I. Ax.* 3

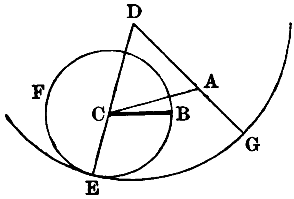


Fig. 4.

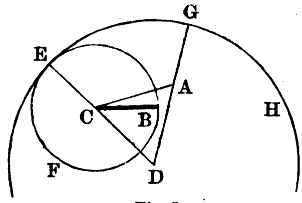


Fig. 5.

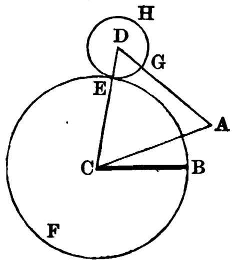


Fig. 6.

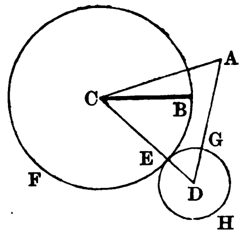


Fig. 7.

PROPOSITION 3.

1. From B draw the straight line $BD = C$; *I.* 2
with centre B and radius BD describe the circle DEF, *Post.* 3
cutting BA at E. BE shall $= C$.
For $BE = BD$, being radii of circle DEF. *I. Def.* 16
But $BD = C$; *Const.*
$\therefore$ $BE = C$. *I. Ax.* 1

2. Let AB and C be the two given straight lines of which AB is the less.
From A draw the straight line $AD = C$; *I.* 2
with centre A and radius AD describe the circle DEF, *Post.* 3
cutting AB produced at E. AE shall $= C$.
For $AE = AD$, being radii of circle DEF, *I. Def.* 16
But $AD = C$; *Const.*
$\therefore$ $AE = C$.

3. Because AD and BE are each $= C$, $\therefore AD = BE$;
$\therefore AB - AD = AB - BE$;
$\therefore$ $BD = AE$.

4. Let AB and C be the two given straight lines.
Produce BA sufficiently far to D, and from AD cut off $AE = C$. *I.* 3

BE is the straight line required.

For $BE = AB + AE = AB + C$.

5. Let AB and C be the two given straight lines, and let AB be the greater.
From AB cut off $AE = C$. *I.* 3

BE is the straight line required.

For $BE = AB - AE = AB - C$.

6. Let AE and EB (fig. 1 to third deduction) be the given straight lines, of which AE is the greater.
Cut off $AD = EB$. *I.* 3

Then $AB = AE + EB$, the sum of AE and EB,

and $DE = AE - EB$, the difference of AE and EB;

$\therefore$ $AB + DE = (AE + EB) + (AE - EB)$,
$= \text{twice } AE$.

7. Repeat the first five lines of the preceding solution, and add
$\therefore$ $AB - DE = (AE + EB) - (AE - EB)$,
$= \text{twice } EB$.
or $AB - DE = AD + EB$,
$= \text{twice } EB$.

PROPOSITION 4.

1. Outside $\angle EDF$. 2. Inside $\angle EDF$.
3. On DF produced. 4. Between D and F.
5. For if $\triangle ABC$ be applied to $\triangle DEF$,
so that B falls on E, and so that BA falls on ED;
then A will coincide with D, because $BA = ED$; *Hyp.*
The rest of the proof is the same as in the text.
For if $\triangle ABC$ be applied to $\triangle DEF$,
so that C falls on F, and so that CA falls on FD;
then A will coincide with D, because $CA = FD$. *Hyp.*
and because CA coincides with FD, and $\angle A = \angle D$, *Hyp.*
$\therefore$ AB will fall on DE.
And because $AB = DE$, *Hyp.*
$\therefore$ B will coincide with E.
The rest of the proof is the same as in the text.

6. Let C be any point in CD (fig. to I. 10).
Join CA, CB.

$$\text{In } \triangle\text{s } ADC, BDC, \left\{\begin{array}{c} AD = BD \\ DC = DC \\ \angle ADC = \angle BDC; \end{array}\right. \qquad \begin{array}{r} \textit{Hyp.} \\ \\ \\ \end{array}$$

$\therefore CA = CB.$ *I.* 4

7. Let E be any point in CD (fig. to I. 10).
Join EA, EB.

$$\text{In } \triangle\text{s } ACE, BCE, \left\{\begin{array}{c} AC = BC \\ CE = CE \\ \angle ACE = \angle BCE; \end{array}\right. \qquad \begin{array}{r} \textit{Hyp.} \\ \\ \textit{Hyp.} \end{array}$$

$\therefore EA = EB.$ *I.* 4

8. Let $\triangle\ CAB$ be isosceles (fig. to I. 10);
and let CD bisect $\angle\ ACB$.

$$\text{In } \triangle\text{s } ACD, BCD, \left\{\begin{array}{c} AC = BC \\ CD = CD \\ \angle ACD = \angle BCD; \end{array}\right. \qquad \begin{array}{r} \textit{Hyp.} \\ \\ \textit{Hyp.} \end{array}$$

$\therefore AD = BD,\ \angle ADC = \angle BDC;$ *I.* 4
$\therefore CD$ is $\perp AB$. *I. Def.* 10

$$9.\ \text{In } \triangle\text{s } ABD, CBD, \left\{\begin{array}{c} AB = CB \\ BD = BD \\ \angle ABD = \angle CBD; \end{array}\right. \qquad \begin{array}{r} \textit{Hyp.} \\ \\ \textit{Hyp.} \end{array}$$

$\therefore AD = CD,\ \angle ADB = \angle CDB.$ *I.* 4

10. Let $ABDC$ be a square (fig. to I. 46).
Join AD, BC.

$$\text{In } \triangle\text{s } CAB, DBA, \left\{\begin{array}{c} CA = DB \\ AB = BA \\ \angle CAB = \angle DBA; \end{array}\right. \qquad \begin{array}{r} \textit{I. Def.}\ 32 \\ \\ \textit{I. Ax.}\ 10 \end{array}$$

$\therefore CB = DA.$ *I.* 4

$$11.\ \text{In } \triangle\text{s } HAE, EBF, \left\{\begin{array}{c} HA = EB \\ AE = BF \\ \angle HAE = \angle EBF; \end{array}\right. \qquad \begin{array}{r} \textit{I. Ax.}\ 7 \\ \textit{I. Ax.}\ 7 \\ \textit{I. Ax.}\ 10 \end{array}$$

$\therefore HE = EF.$ *I.* 4
Hence also $EF = FG$, $FG = GH$;
$\therefore EFGH$ has all its sides equal.

12. Let $ABCD$, $EFGH$ be squares described on the equal straight lines AB, EF.

For if square $ABCD$ be applied to square $EFGH$,
so that A falls on E, and so that AB falls on EF;
then B will coincide with F, because $AB = EF$. *Hyp.*

And because AB coincides with EF, and $\angle B = \angle F$, *I. Ax.* 10

$\therefore$ BC will fall on FG.

And because $BC = FG$,

$\therefore$ C will coincide with G.

Again, because BC coincides with FG,

and $\angle C = \angle G$, *I. Ax.* 10

$\therefore$ CD will fall on GH.

And because $CD = GH$,

$\therefore$ D will coincide with H.

Lastly, because A coincides with E, and D with H,

$\therefore$ AD will coincide with EH.

Hence $ABCD$ coincides with $EFGH$, and is equal to it.

13. Let $ABCD$, $EFGH$ be two quadrilaterals, having $AB = EF$, $BC = FG$, $CD = GH$, $\angle B = \angle F$, $\angle C = \angle G$.

For if quadrilateral $ABCD$ be applied to quadrilateral $EFGH$, &c.

The rest of the proof is identical with that of the twelfth deduction.

PROPOSITION 5.

1. For if the sides opposite the angles were equal,
the angles themselves would be equal; *I.* 5
which is not the case.

2. Because $AB = AC$, $\therefore \angle ABC = \angle ACB$; *I.* 5
because $DB = DC$, $\therefore \angle DBC = \angle DCB$; *I.* 5
$\therefore \angle ABC + \angle DBC = \angle ACB + \angle DCB$,
that is, $\angle ABD = \angle ACD$.

3. Because $AB = AC$, $\therefore \angle ABC = \angle ACB$; *I.* 5
because $DB = DC$, $\therefore \angle DBC = \angle DCB$; *I.* 5
$\therefore \angle ABC - \angle DBC = \angle ACB - \angle DCB$,
or $\angle DBC - \angle ABC = \angle DCB - \angle ACB$,
that is, $\angle ABD = \angle ACD$.

4. It has been proved in the second deduction that $\angle ABD = \angle ACD$.

Now in $\triangle$s ABD, ACD, $\left\{\begin{array}{ll} AB = AC & \textit{Hyp.} \\ BD = CD & \textit{Hyp.} \\ \angle ABD = \angle ACD; & \end{array}\right.$

$\therefore \angle BAD = \angle CAD$, $\angle BDA = \angle CDA$. *I.* 4

5. In $\triangle$s ABE, ACD, $\left\{\begin{array}{ll} BA = CA & \textit{Hyp.} \\ AE = AD & \textit{Hyp.} \\ \angle BAE = \angle CAD; & \end{array}\right.$

$\therefore$ $\triangle$s ABE, ACD are equal in all respects. *I.* 4

Because $AB = AC$, and $AD = AE$; *Hyp.*

$\therefore$ $DB = EC$. *I. Ax.* 3

In $\triangle$s DBC, ECB, $\left\{\begin{array}{ll} DB = EC & \\ BC = CB & \\ \angle DBC = \angle ECB; & \textit{I. } 5 \end{array}\right.$

$\therefore$ $\triangle$s DBC, ECB are equal in all respects.

6. Let $ABCD$ be a rhombus; join BD.

Because $AB = AD$, $\therefore$ $\angle ABD = \angle ADB$; *I.* 5

because $CB = CD$, $\therefore$ $\angle CBD = \angle CDB$; *I.* 5

$\therefore$ $\angle ABD + \angle CBD = \angle ADB + \angle CDB$,

that is, $\angle ABC = \angle ADC$.

But $\angle ABC$ and $\angle ADC$ are any pair of opposite angles;

$\therefore$ the other pair of opposite angles are equal.

7. Join OA, OB, OC.

In $\triangle$s BDO, CDO, $\left\{\begin{array}{ll} BD = CD & \textit{Hyp.} \\ DO = DO & \\ \angle BDO = \angle CDO; & \textit{I. Ax. } 10 \end{array}\right.$

$\therefore$ $OB = OC$. *I.* 4

Similarly, $OC = OA$; $\therefore$ $OA = OB$;

$\therefore$ $\angle OAB = \angle OBA$. *I.* 5

8. If $\triangle ABC$ be applied to its trace, so that A may fall on A, and so that AC may fall along AB;

then C will coincide with B, because $AC = AB$. *Hyp.*

And because $\angle A$ remains the same, AB will fall along AC;

and because $AB = AC$, $\therefore$ B will coincide with C.

Hence CB will coincide with BC,

and $\angle ACB$ will coincide with $\angle ABC$;

$\therefore$ $\angle ACB = \angle ABC$.

Again, since AC coincides with AB, CE will fall along BD,

and since AB coincides with AC, BD will fall along CE;

$\therefore$ $\angle BCE$ will coincide with $\angle CBD$;

$\therefore$ $\angle BCE = \angle CBD$.

9. Let $\triangle ABC$ be isosceles, and the vertical $\angle A$ bisected by AD.

In $\triangle$s BAD, CAD, $\left\{\begin{array}{ll} BA = CA & \textit{Hyp.} \\ AD = AD & \\ \angle BAD = \angle CAD; & \textit{Hyp.} \end{array}\right.$

$\therefore$ $\angle ABD = \angle ACD$. *I.* 4

PROPOSITION 6.

1. For if the angles opposite the sides were equal,
   the sides themselves would be equal; *I.* 6
   which is not the case.
2. Because $\angle\ ABC = \angle\ ACB$, *I.* 5
   $\therefore\ \angle\ DBC = \angle\ DCB$; *I. Ax.* 7
   $\therefore\ DB = DC$, and $\triangle\ DBC$ isosceles. *I.* 6
3. It was proved in I. 5 that $\angle\ GBC = \angle\ FCB$,
   that is, $\angle\ HBC = \angle\ HCB$;
   $\therefore\ HB = HC$, and $\triangle\ HBC$ isosceles. *I.* 6
4. It was proved in I. 5 that $FC = GB$,
   and in the third deduction that $HC = HB$;
   $\therefore\ FH = GH$. *I. Ax.* 3

   In $\triangle$s AFH, AGH, $\left\{\begin{array}{l} AF = AG \\ FH = GH \\ \angle\ AFH = \angle\ AGH; \end{array}\right.$ *Const.* ... *I.* 5

   $\therefore\ \angle\ FAH = \angle\ GAH$. *I.* 4
5. Let $\angle\ DAE$ be the given angle (fig. to I. 5).
   In AD take any two points B and F;
   from AE cut off $AC = AB$, $AG = AF$; *I.* 3
   join BG, CF intersecting at H.
   AH will bisect $\angle\ DAE$.
6. If $\triangle\ ABC$ be applied to its trace, so that C may fall on B,
   and so that CB may fall along BC;
   then B will fall on C, because CB remains the same.
   And because $\angle\ BCA = \angle\ CBA$;
   $\therefore\ CA$ will fall along BA, and BA along CA.
   Hence CA will coincide with BA, and BA with CA;
   $\therefore\ BA = CA$.

PROPOSITION 7.

1. If possible, let CAB, DAB (fig. to I. 7), be two equilateral triangles on the same side of the same base AB.
   Then, since $AC = AB$, and $AD = AB$,
   $\therefore\ AC = AD$.
   Similarly, $BC = BD$, which is impossible. *I.* 7

2. If possible, let CAB, DAB (fig. to I. 7), be two isosceles triangles on the same side of the same base AB, and having their sides each = a given straight line M.
Then, since $AC = M$, and $AD = M$;
$\therefore \quad AC = AD.$
Similarly, $BC = BD$, which is impossible. *I.* 7

3. Let A and B be the centres of two circles, and, if possible, let the circles cut each other at C and D above their line of centres AB.
Join AC, AD, BC, BD.
Then $AC = AD$, being radii of the same circle,
and $BC = BD$, being radii of the same circle;
which is impossible. *I.* 7
Hence also the circles cannot cut each other at more than one point below their line of centres AB.

PROPOSITION 8.

1. Let $\triangle\ ABC$ be isosceles, D the middle point of the base BC.

$$\text{In } \triangle\text{s } BAD, CAD, \begin{cases} BA = CA & \textit{Hyp.} \\ AD = AD \\ BD = CD; & \textit{Hyp.} \end{cases}$$

$\therefore \angle BAD = \angle CAD$, and $\angle ADB = \angle ADC$; *I.* 8
$\therefore \angle$s ADB, ADC are right. *I. Def.* 10

2. Let $ABCD$ be a rhombus. Join BD.

$$\text{In } \triangle\text{s } BAD, BCD, \begin{cases} BA = BC \\ AD = CD \\ BD = BD; \end{cases}$$

$\therefore \angle A = \angle C.$ *I.* 8
Hence also $\angle ABC = \angle ADC$.

3. Let $ABCD$ be a rhombus. Join BD.

$$\text{In } \triangle\text{s } BAD, BCD, \begin{cases} BA = BC \\ AD = CD \\ BD = BD; \end{cases}$$

$\therefore \angle ABD = \angle CBD$, and $\angle ADB = \angle CDB$. *I.* 8

4. In $\triangle$s BAD, BCD,

$$\begin{cases} BA = BC \\ AD = CD \\ BD = BD; \end{cases}$$

$\therefore \angle A = \angle C$, $\angle ABD = \angle CBD$, $\angle ADB = \angle CDB$. *I.* 8

5. Let ABC, DBC be the two isosceles triangles, standing on opposite sides of the same base BC, and let AD be joined.

In $\triangle$s BAD, CAD, $\begin{cases} BA = CA \\ AD = AD \\ BD = CD; \end{cases}$

$\therefore \angle BAD = \angle CAD$, and $\angle BDA = \angle CDA$. *I.* 8

6. Let ABC, DBC be the two isosceles triangles, standing on the same side of the same base BC, and let AD be joined and produced to meet BC at E.

Suppose D to fall inside $\triangle$ ABC.

In $\triangle$s BAD, CAD, $\begin{cases} BA = CA \\ AD = AD \\ BD = CD; \end{cases}$

$\therefore \angle BAD = \angle CAD$. *I.* 8

In $\triangle$s BAE, CAE, $\begin{cases} BA = CA \\ AE = AE \\ \angle BAE = \angle CAE; \end{cases}$

$\therefore BE = CE$. *I.* 4

In $\triangle$s BDE, CDE, $\begin{cases} BD = CD \\ DE = DE \\ BE = CE; \end{cases}$

$\therefore \angle BDE = \angle CDE$. *I.* 8

7. Let E be the point where AD or AD produced cuts BC.

Repeat the previous solution, and at the end

for $\angle BDE = \angle CDE$,

substitute $\angle BED = \angle CED$; and they are adjacent;

$\therefore$ each is right.

8. On the given straight line as base describe two isosceles triangles; the straight line joining their vertices, or that straight line produced will bisect the given straight line.

9. Let $ABCD$ be a rhombus or a square, AC and BD the two diagonals.

ABD, CBD are two isosceles triangles on the same base BD,

$\therefore$ by the seventh deduction from I. 8, AC bisects BD perpendicularly.

Hence also BD bisects AC perpendicularly.

10. This is equivalent to the seventh deduction from I. 8.

11. In $\triangle$s ABC, DEF let $AB = DE$, $AC = DF$, $BC = EF$.

Suppose $\triangle$ ABC to be lifted up, and placed so that B falls on E, and so that BC falls along EF;

then C will coincide with F, because $BC = EF$. *Hyp.*

Let, however, $\triangle\ ABC$ fall on the opposite side of EF from $\triangle\ DEF$, and let G be the point where the vertex A falls; then $\triangle\ GEF$ will represent $\triangle\ ABC$. Join DG.

Because $EG = ED$, $\therefore \angle\ EGD = \angle\ EDG$, *I.* 5
and because $FG = FD$, $\therefore \angle\ FGD = \angle\ FDG$; *I.* 5
$\therefore \angle\ EGD + \angle\ FGD = \angle\ EDG + \angle\ FDG$;
$\therefore \angle\ EGF = \angle\ EDF$, that is, $\angle\ BAC = \angle\ EDF$.
Hence also $\angle\ ABC = \angle\ DEF$, and $\angle\ ACB = \angle\ DFE$.

PROPOSITION 9.

1. In the proposition, $\triangle$s DCF, ECF were proved equal in all respects;
   $\therefore \angle\ DFC = \angle\ EFC$.
2. F might fall inside $\triangle\ DCE$; it might fall outside $\triangle\ DCE$; it might fall on C.
3. In the first position the demonstration remains the same as in the text, as may be seen by constructing the figure.
4. Yes; because no use is made of the equality of the three sides of $\triangle\ DEF$, but only of the equality of two of them, DF and EF.
5. Because $CD = CE$, $\therefore \angle\ CDE = \angle\ CED$; *I.* 5
   and because $FD = FE$, $\therefore \angle\ FDE = \angle\ FED$; *I.* 5
   $\therefore$ the whole $\angle\ CDF =$ the whole $\angle\ CEF$.

   In $\triangle$s CDF, CEF, $\left\{\begin{array}{l} CD = CE \\ DF = EF \\ \angle\ CDF = \angle\ CEF. \end{array}\right.$

   $\therefore \angle\ DCF = \angle\ ECF$, and $\angle\ DFC = \angle\ EFC$. *I.* 4
6. Bisect the angle, and then bisect the two halves of the angle.
7. By continued bisections the number of parts into which an angle may be divided is extended to 8, 16, 32, 64, 128, &c.
8. Let ABC be an equilateral triangle, and let AD bisect the vertical angle BAC, and meet BC at D.
   It may be proved, as in the eighth deduction from *I.* 4, that AD is $\perp$ BC and bisects BC.
   Hence $\triangle\ ADB$ is right-angled, and $BD =$ half of BC
   $=$ half of AB. And since $\angle\ BAD$ is half of $\angle\ BAC$,
   it is also half of $\angle\ ABD$. *I.* 5, *Cor.*

PROPOSITION 10.

1. Yes; because no use is made of the equality of the three sides of $\triangle$ ABC, but only of the equality of two of them, AC and BC.
2. In the proposition, $\triangle$s ACD, BCD were proved equal in all respects; $\therefore$ $\angle ADC = \angle BDC$.
   Now these are adjacent angles; $\therefore$ CD is $\perp$ AB.
3. The figure $ACBF$ is a rhombus; and it is proved in the ninth deduction from I. 8 that the diagonals bisect each other.
4. On the given straight line as base and on opposite sides of it construct two equilateral triangles; and join their vertices.

5, 6. See the ninth deduction from I. 8.

7. Bisect the straight line, and then bisect the halves of it.
8. By continued bisections, the number of equal parts is extended to 8, 16, 32, 64, 128, &c.
9. Let AB be the given straight line.
   Produce AB its own length to C, and bisect BC at D.
   AD is half as long again as AB.
10. Let BA and BC be the two given straight lines.
    Place BA and BC in the same straight line, but so that they are measured in opposite directions from B; and bisect AC.
11. Let BA and BC be the two given straight lines.
    Place BA and BC in the same straight line, but so that they are measured in the same direction from B; and bisect AC.
12. By the eighth deduction from I. 4, or by reasoning similar to that in the proposition, BC is bisected at F.
    Now since $AB = BC$, $\quad \therefore BD = BF$.

$$\text{In } \triangle\text{s } ABF, CBD, \left\{ \begin{array}{l} AB = CB \\ BF = BD \\ \angle B = \angle B; \end{array} \right.$$

$\therefore AF = CD$. *I.* 4

PROPOSITION 11.

1. Yes; because no use is made of the equality of the three sides of $\triangle$ DEF, but only of the equality of two of them, DF and EF.
2. It would be necessary to produce the given straight line.

3. At the given point draw a perpendicular to the given straight line, and bisect either of the right angles so formed.
4. Let AB be the given straight line (fig. to I. 11), and C the given point in it.
   At C draw $CF \perp AB$; *I.* 11
   bisect $\angle\ BCF$ by CG, and $\angle\ BCG$ by CH. *I.* 9
   Then $\angle\ BCH$ is one-fourth of a right angle.
5. Take any straight line AB, and at B draw $BC \perp AB$ and $= AB$. Join AC.
6. Take any straight line AB, and at B draw $BC \perp AB$.
   With A as centre and a radius = twice AB, describe a circle cutting BC at C. Join AC.
7. Let AB be the given straight line, C and D the given points.
   Join CD, and bisect it at E. *I.* 10
   From E draw $EF \perp CD$, and meeting AB or AB produced at F. *I.* 11
   Join CF, DF.

$$\text{In } \triangle\text{s } CEF,\ DEF, \left\{ \begin{array}{c} CE = DE \\ EF = EF \\ \angle\ CEF = \angle\ DEF; \end{array} \right.$$

   $\therefore CF = DF$. *I.* 4
   The problem is impossible when EF does not meet AB, and that will occur when CD or CD produced is $\perp AB$.

$$\text{8. In } \triangle\text{s } ALO,\ BLO, \left\{ \begin{array}{c} AL = BL \\ LO = LO \\ \angle\ ALO = \angle\ BLO; \end{array} \right.$$

   $\therefore \quad OA = OB$. *I.* 4
   Hence also $OA = OC$; $\therefore OA = OB = OC$.
9. Proposition 9 becomes Proposition 11 when AC and CB, the arms of the angle ACB, are in the same straight line.

PROPOSITION 12.

1. Not necessarily.
2. In the proposition, $\triangle$s CGE, CGF were proved equal in all respects; $\therefore \angle\ ECG = \angle\ FCG$.
3. Yes; because then

$$\text{in } \triangle\text{s } CGE,\ CGF, \left\{ \begin{array}{c} EC = FC \\ CG = CG \\ \angle\ ECG = \angle\ FCG; \end{array} \right.$$

$\therefore \angle CGE = \angle CGF$; *I.* 4

$\therefore$ CG is $\perp AB$.

4. It would answer equally well if the circle described with C as centre and CD as radius cut AB or AB produced in two points.

5. Let AB be the given straight line, C and D the given points.
   From C draw $CE \perp AB$; *I.* 12
   and produce CE its own length to F.
   Join DF and produce it to meet AB at G, and join CG.

   In $\triangle$s CEG, FEG, $\left\{\begin{array}{c} CE = FE \\ EG = EG \\ \angle CEG = \angle FEG; \end{array}\right.$

   $\therefore \angle CGE = \angle FGE$. *I.* 4

   The problem is impossible if DF produced does not meet AB.

6. Let AB be the given straight line, C the given point without it.
   In AB take any two points D and E.
   With D as centre and DC as radius, describe a circle;
   with E as centre and EC as radius describe a circle cutting the former again at F. Join CF. CF is $\perp AB$.

PROPOSITION 13.

1. AED, CEB; AEC, DEB; BEC, DEA; BED, CEA.
2. BOE, BOA, COA, DOA, DOE.
3. CBD, BCE, ABC, BCA, CFD, BGE, GBD, FCE.
4. COB, DOB, DOA, COA.
5. Because $\angle AOC = \angle BOD$;

   $\therefore \angle AOC + \angle COD = \angle BOD + \angle COD$;

   $\therefore \angle AOD = \angle BOC$.

   Again because $\angle AOD = \angle BOC$;

   $\therefore \angle AOD - \angle COD = \angle BOC - \angle COD$;

   $\therefore \angle AOC = \angle BOD$.

6. Let BF bisect $\angle ABD$, and BG bisect $\angle ABC$.

   Then $\angle ABF + \angle ABG =$ half of $(\angle ABC + \angle ABD)$;

   $\therefore \angle FBG =$ half of 2 rt. $\angle$s, *I.* 13

   $=$ a right angle.

7. The angles on the other side of the base are supplementary to the angles at the base; and since the angles at the base are equal, the angles on the other side of the base must be equal also.

8. The angles at the base are equal; *I.* 5
and the exterior angles are the supplements of the angles at the base;
$\therefore$ the exterior angles are equal.

9. Since $\angle\ DBC = \angle\ ECB$;
$\therefore$ the supplement of $\angle\ DBC =$ the supplement of $\angle\ ECB$;
$\therefore\ \angle\ ABC = \angle\ ACB$;
$\therefore\ \triangle\ ABC$ is isosceles. *I.* 6

10. Since the exterior angles are equal, their supplements, namely the angles at the base, are also equal;
$\therefore$ the triangle is isosceles. *I.* 6

PROPOSITION 14.

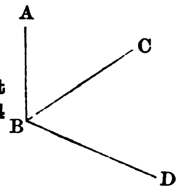
1. If the square $EFGH$ fall on the same side of CD as the square $ABCD$, then FG will fall along CB, because $\angle\ EFG = \angle\ DCB$. If the square $EFGH$ fall on the opposite side of CD from the square $ABCD$, then FG will be in the same straight line as CB, because the two adjacent angles at C
$= 2$ rt. $\angle$s. *I.* 14

2. Because $\angle\ AEC = \angle\ BED$;
$\therefore\ \angle\ AEC + \angle\ BEC = \angle\ BED + \angle\ BEC.$
But $\angle\ AEC + \angle\ BEC = 2$ rt. $\angle$s; *I.* 13
$\therefore\ \angle\ BED + \angle\ BEC = 2$ rt. $\angle$s;
$\therefore\ EC$ and ED are in the same straight line. *I.* 14

3. Because $\angle\ AEC = \angle\ BED$,
and $\angle\ AED = \angle\ BEC$;
$\therefore\ \angle\ AEC + \angle\ AED = \angle\ BED + \angle\ BEC.$
But $\angle\ AEC + \angle\ AED + \angle\ BED + \angle\ BEC = 4$ rt. $\angle$s;
I. 13, *Cor.* 2
$\therefore\ \angle\ AEC + \angle\ AED = 2$ rt. $\angle$s;
$\therefore\ CE$ and ED are in the same straight line. *I.* 14
Hence also AE and EB are in the same straight line.

4. In $\triangle$s OMP, OMQ, $\left\{\begin{array}{l} PM = QM \\ MO = MO \\ \angle\ PMO = \angle\ QMO; \end{array}\right.$
$\therefore\ \angle\ POM = \angle\ QOM.$ *I.* 4
Hence also $\angle\ PON = \angle\ RON$;
$\therefore\ \angle\ POM + \angle\ PON = \angle\ QOM + \angle\ RON.$

But $\angle POM + \angle PON =$ a right angle;
$\therefore \angle POM + \angle PON + \angle QOM + \angle RON = 2$ rt. $\angle$s;
$\therefore QO$ and OR are in the same straight line. *I.* 14



5. No.

PROPOSITION 15.

1. Because $\angle AEC + \angle AED = 2$ rt. $\angle$s, *I.* 13
and $\angle BED + \angle AED = 2$ rt. $\angle$s; *I.* 13
$\therefore \angle AEC + \angle AED = \angle BED + \angle AED$;
$\therefore \angle AEC = \angle BED$.
2. Because $\angle BEC + \angle AEC = 2$ rt. $\angle$s, *I.* 13
and $\angle AED + \angle AEC = 2$ rt. $\angle$s; *I.* 13
$\therefore \angle BEC + \angle AEC = \angle AED + \angle AEC$;
$\therefore \angle BEC = \angle AED$.
3. Because $\angle BEC + \angle BED = 2$ rt. $\angle$s, *I.* 13
and $\angle AED + \angle BED = 2$ rt. $\angle$s; *I.* 13
$\therefore \angle BEC + \angle BED = \angle AED + \angle BED$;
$\therefore \angle BEC = \angle AED$.
4. Because $\angle AEF = \angle BEG$, *I.* 15
and $\angle DEF = \angle CEG$; *I.* 15
$\therefore \angle BEG = \angle CEG$.
5. Because $\angle AED = \angle BEC$; *I.* 15
$\therefore$ half of $\angle AED =$ half of $\angle BEC$;
$\therefore \angle AEF = \angle BEG$;
$\therefore FE$ and EG are in the same straight line by the second deduction from I. 14.
6. If ED be not in the same straight line as CE, produce CE to F.
Then $\angle AEC = \angle BEF$. *I.* 15
But $\angle AEC = \angle BED$;
$\therefore \angle BEF = \angle BED$, which is impossible.
Hence ED is in the same straight line as CE.
7. Produce AF to meet BC at G.
Since $\angle FBC = \angle FCB$, $\therefore FB = FC$. *I.* 6
Since $\angle AFE = \angle CFG$, and $\angle AFD = \angle BFG$; *I.* 15
$\therefore \angle BFG = \angle CFG$.

In $\triangle$s BFG, CFG, $\left\{\begin{array}{c} BF = CF \\ FG = FG \\ \angle\ BFG = \angle\ CFG\,; \end{array}\right.$

$\therefore$ $BG = CG$, and $\angle\ BGF = \angle\ CGF$. *I.* 4

In $\triangle$s BGA, CGA, $\left\{\begin{array}{c} BG = CG \\ GA = GA \\ \angle\ BGA = \angle\ CGA\,; \end{array}\right.$

$\therefore$ $AB = AC$, and $\triangle$ ABC is isosceles. *I.* 4

PROPOSITION 16.

1. Because ABE is a triangle, $\angle\ AEF$ or $\angle\ BEC$ is greater than $\angle\ A$.

   Because ABC is a triangle, $\angle\ ACD$ or $\angle\ BCG$ is greater than $\angle\ A$.

2. Because FBC is a triangle, $\angle\ FCD$ is greater than $\angle\ F$.

   Because FEC is a triangle, $\angle\ FCG$ is greater than $\angle\ F$.

   Because FEC is a triangle, $\angle\ BEC$ or $\angle\ AEF$ is greater than $\angle\ F$.

3. Because ABE is a triangle, $\angle\ AEF$ or $\angle\ BEC$ is greater than $\angle\ ABE$.

   Because ABC is a triangle, $\angle\ ACD$ or $\angle\ BCG$ is greater than $\angle\ ABC$;

   and therefore greater than $\angle\ ABE$.

4. Because EBC is a triangle, $\angle\ ACD$ or $\angle\ BCG$ is greater than $\angle\ CBE$.

   Because EBC is a triangle, $\angle\ AEB$ or $\angle\ CEF$ is greater than $\angle\ CBE$.

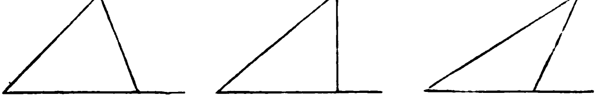
5. Because EBC is a triangle, $\angle\ AEB$ or $\angle\ CEF$ is greater than $\angle\ ECB$.

6. Because EBC is a triangle, $\angle\ ECD$ or $\angle\ BCG$ is greater than $\angle\ BEC$.

7. This is the same as the fifth deduction, since $\angle\ BCE$ is the same as $\angle\ ACB$.

8. Because FEC is a triangle, $\angle\ AEF$ or $\angle\ BEC$ is greater than $\angle\ ECF$.

9.



10. If possible, from C let there be drawn to the given straight line AB two perpendiculars CD, CE.

Then CDE is a triangle, and the exterior $\angle\ CEB$ is greater than the interior opposite $\angle\ CDE$; *I.* 16

which is impossible, since they are both right angles.

11. Because BD, a side of $\triangle\ ABD$, is produced to C;

$\therefore\ \angle\ ADC$ is greater than $\angle\ BAD$. *I.* 16

But $\angle\ BAD = \angle\ DAC$;

$\therefore\ \angle\ ADC$ is greater than $\angle\ DAC$.

Hence also $\angle\ ADB$ is greater than $\angle\ BAD$.

12. (1) In $\triangle$s AEF, CEB, $\begin{cases} AE = CE \\ EF = EB \\ \angle\ AEF = \angle\ CEB; \end{cases}$

$\therefore\ AF = CB$. *I.* 4

(2) In the proposition, $\triangle$s AEB, CEF were proved equal;

$\therefore\ \triangle\ AEB + \triangle\ BEC = \triangle\ CEF + \triangle\ BEC$;

$\therefore\ \triangle\ ABC = \triangle\ BCF$.

(3) In the proposition, $\triangle$s AEB, CEF were proved equal;

$\therefore\ \triangle\ AEB + \triangle\ AEF = \triangle\ CEF + \triangle\ AEF$;

$\therefore\ \triangle\ ABF = \triangle\ ACF$.

13. Let ABC be any triangle on the base BC (fig. to I. 16).

Bisect AC at E; join BE, and produce it its own length to F, and join FC.

Then, by the preceding deduction, $\triangle\ ABC = \triangle\ FBC$ and $AF = BC$.

Bisect CF at H; join BH and produce it its own length to K, and join KC.

Then by the preceding deduction, $\triangle\ FBC = \triangle\ KBC$, and $FK = BC$.

Hence $\triangle\ ABC = \triangle\ FBC = \triangle\ KBC$, and $AF = FK$.

Again bisect CK at L; join BL, and so on.

14. If possible, from the given point C let there be drawn to the given straight line AB, three equal straight lines CD, CE, CF, CE lying between CD and CF.

Because DE, a side of $\triangle\ CDE$, is produced to F,

$\therefore\ \angle\ CEF$ is greater than $\angle\ CDE$. *I.* 16

Because $CD = CF$, $\therefore\ \angle\ CDE = \angle\ CFE$; *I.* 5

$\therefore\ \angle\ CEF$ is greater than $\angle\ CFE$.

But because $CE = CF$, $\therefore\ \angle\ CEF = \angle\ CFE$; *I.* 5

which is impossible.

Hence from C there cannot be drawn to AB more than two equal straight lines.

15. Let ABC be a triangle, and let AB be produced to D, and BC to E.

Then $\angle\ CBD$ is greater than $\angle\ ACB$, *I.* 16

and $\angle\ ACE$ is greater than $\angle\ ABC$; *I.* 16

$\therefore\ \angle\ CBD + \angle\ ACE$ is greater than $\angle\ ACB + \angle\ ABC$.

But $\angle\ CBD + \angle\ ACE + \angle\ ACB + \angle\ ABC = 4$ rt. $\angle$s; *I.* 13

$\therefore\ \angle\ CBD + \angle\ ACE$ is greater than the half of 4 rt. $\angle$s, that is, greater than 2 rt. $\angle$s.

PROPOSITION 17.

1. If there were two right angles, the sum of these two would not be less than two right angles, as is necessary by I. 17.
   Similarly, if there were two obtuse angles, or one right and one obtuse angle.
2. This is the preceding deduction put in other words.
3. If there could be two, they would with the given straight line form a triangle, two of whose angles were not less than two right angles.
4. Let ABC be a triangle. Take any point D in BC and join AD.
   Then $\angle\ ADC$ is greater than $\angle\ B$, *I.* 16
   and $\angle\ ADB$ is greater than $\angle\ C$; *I.* 16
   $\therefore\ \angle\ ADC + \angle\ ADB$ is greater than $\angle\ B + \angle\ C$.
   But $\angle\ ADC + \angle\ ADB = 2$ rt. $\angle$s; *I.* 13
   $\therefore\ \angle\ B + \angle\ C$ is less than 2 rt. $\angle$s.
   Now $\angle\ B$ and $\angle\ C$ are any pair of the angles of the triangle.
5. Since the angles at the base of an isosceles triangle are equal, and since the two together are less than two right angles;
   $\therefore$ each must be less than the half of two right angles;
   $\therefore$ each must be less than a right angle.
6. Since any pair of the angles of an equilateral triangle are equal;
   $\therefore$ by the preceding deduction, each of this pair must be acute; $\therefore$ all are acute.
7. If the less were a right angle, then the greater would be greater than a right angle; and the sum of the two would not be less than two right angles.
   If the less were obtuse, then the greater would be obtuse; and the sum of the two would not be less than two right angles.

8. Let A, B, C denote the three interior angles of any triangle.
Then $A + B$ is less than 2 rt. $\angle$s, *I.* 17
$B + C$ is less than 2 rt. $\angle$s, *I.* 17
$C + A$ is less than 2 rt. $\angle$s; *I.* 17
$\therefore$ $2A + 2B + 2C$ is less than 6 rt. $\angle$s;
$\therefore$ $A + B + C$ is less than 3 rt. $\angle$s.

9. Let A, B, C denote the three interior angles of any triangle, A', B', C' the three exterior angles adjacent to A, B, C.
Then $A + A' + B + B' + C + C' = 6$ rt. $\angle$s. *I.* 13
But $A + B + C$ is less than 3 rt. $\angle$s, by the preceding deduction;
$\therefore$ $A' + B' + C'$ is greater than 3 rt. $\angle$s.

10. Let ABC be a triangle, right-angled at A, and let AD be drawn $\perp$ BC. AD must fall inside the triangle.
If not, let AD meet BC produced at D.
Since $\angle$ BAC is right, $\therefore$ $\angle$ ACB is acute. *I.* 17
But $\angle$ ACB is greater than $\angle$ ADC; *I.* 16
that is, an acute angle is greater than a right angle, which is impossible.

11. Let ABC be a triangle, obtuse-angled at A, and let AD be drawn $\perp$ BC. AD must fall inside the triangle.
If not, let AD meet BC produced at D.
Since $\angle$ BAC is obtuse, $\therefore$ $\angle$ ACB is acute. *I.* 17
But $\angle$ ACB is greater than $\angle$ ADC; *I.* 16
that is, an acute angle is greater than a right angle, which is impossible.

12. Let ABC be an acute-angled triangle, and let AD be drawn $\perp$ BC. AD must fall inside the triangle.
If not, let AD meet BC produced at D.
Then $\angle$ ACB is greater than $\angle$ ADC. *I.* 16
But $\angle$ ACB is acute;
$\therefore$ an acute angle is greater than a right angle, which is impossible.

13. Let ABC be a triangle, obtuse-angled at C, and let AD be drawn $\perp$ BC. AD must fall outside the triangle.
If not, let AD meet BC at D.
Then $\angle$ ADB is greater than $\angle$ ACD; *I.* 16
that is, a right angle is greater than an obtuse angle, which is impossible.

PROPOSITION 18.

1. For if the sides opposite them were unequal, the angles would be unequal, which they are not.
2. For if any pair of its angles were equal, that pair of sides opposite to them would also be equal, which they are not.
3. For, of the two angles, that opposite the less side is the less;
   $\therefore$ by the seventh deduction from I. 17, it must be acute.
4. Join BD.
   Because AD is greater than AB,
   $\therefore$ $\angle ABD$ is greater than $\angle ADB$. *I.* 18
   Because CD is greater than CB,
   $\therefore$ $\angle CBD$ is greater than $\angle CDB$; *I.* 18
   $\therefore$ $\angle ABD + \angle CBD$ is greater than $\angle ADB + \angle CDB$;
   $\therefore$ $\angle ABC$ is greater than $\angle ADC$.
   By joining AC, $\angle BCD$ may in like manner be proved greater than $\angle BAD$.
5. $\angle ABC$ is greater than $\angle D$; *I.* 16
   $\therefore$ $\angle ABC$ is greater than $\angle ACD$, which $= \angle D$. *I.* 5
   Much more, then, is $\angle ABC$ greater than $\angle ACB$.
6. In $\triangle$s ABD, AED, $\left\{\begin{array}{c} AB = AE \\ AD = AD \\ \angle BAD = \angle EAD; \end{array}\right.$
   $\therefore$ $\angle ABD = \angle AED$. *I.* 4
   But $\angle AED$ is greater than $\angle C$; *I.* 16
   $\therefore$ $\angle ABD$ is greater than $\angle C$.

PROPOSITION 19.

1. Let ABC be a triangle, right-angled at C.
   Then AC is less than AB. *I.* 19, *Cor.*
2. This is equivalent to the preceding deduction.
3. Let ABC be a triangle, obtuse-angled at C.
   Since $\angle B + \angle C$ is less than 2 rt. $\angle$s, *I.* 17
   and $\angle C$ is obtuse; $\therefore$ $\angle B$ is acute;
   $\therefore$ AB is greater than AC. *I.* 19
4. Draw the diagonal AC.
   Then $\angle ACE$ is obtuse, since it is greater than $\angle BCE$;
   $\therefore$ AE is greater than AC, by the preceding deduction.

5. If possible, from the given point C let there be drawn to the given straight line AB, three equal straight lines CD, CE, CF, CE lying between CD and CF.
   Then $\angle\ CEF$ is greater than $\angle\ CDE$. *I.* 16
   But $\angle\ CDE = \angle\ CFE$, since $CD = CF$; *I.* 5
   $\therefore$ $\angle\ CEF$ is greater than $\angle\ CFE$;
   $\therefore$ CF is greater than CE. *I.* 19
   Now CE was supposed $=$ CF.
6. For if it could, then there could be drawn to the given straight line from the centre of the circle more than two equal straight lines; which is impossible, by the preceding deduction.
7. By the eleventh deduction from I. 16, $\angle\ ADC$ is greater than $\angle\ DAC$, and $\angle\ ADB$ greater than $\angle\ BAD$;
   $\therefore$ AC is greater than CD, and AB greater than BD. *I.* 19

PROPOSITION 20.

1. In connection with the requisite figure, write out, word for word, the construction and proof given in the text; but wherever B occurs substitute C, and wherever C occurs substitute B.
2. Let ABC be a triangle, and let AD be drawn $\perp$ BC.
   Then AB is greater than BD, and AC greater than CD, by the first deduction from I. 19;
   $\therefore$ $AB + AC$ is greater than $BD + CD$, or BC.
   If AD falls outside the triangle, the proof still holds, only instead of BC being equal to the sum of BD and CD, it is equal to their difference.
3. In the seventh deduction from I. 19, it is proved that AB is greater than BD, and AC greater than CD;
   $\therefore$ $AB + AC$ is greater than $BD + CD$, or BC.
4. Let AD and BC intersect at O.
   Then $AO + OC$ is greater than AC,
   and $BO + OD$ is greater than BD; *I.* 20
   $\therefore$ $AO + OD + BO + OC$ is greater than $AC + BD$;
   $\therefore$ $AD + BC$ is greater than $AC + BD$.
5. Let ABC be a circle whose centre is O, and let AB be a diameter, and CD a straight line in the circle which is not a diameter.
   Join CO and produce it to meet the circle at E, and join OD.

Then $CO + OD$ is greater than CD; *I.* 20

$\therefore$ $CO + OE$ is greater than CD.

But $CO + OE = AO + OB$, or AB;

$\therefore$ AB is greater than CD.

6. Let $ABCD$ be a quadrilateral. Join BD.

Then $BC + CD$ is greater than BD; *I.* 20

$\therefore$ $AB + BC + CD$ is greater than $AB + BD$.

Now $AB + BD$ is greater than AD;

$\therefore$ $AB + BC + CD$ is greater than AD.

7. Let $ABCDE$ be a polygon. Join BE.

Then $BC + CD + DE$ is greater than BE, by the preceding deduction;

$\therefore$ $AB + BC + CD + DE$ is greater than $AB + BE$.

Now $AB + BE$ is greater than AE; *I.* 20

$\therefore$ $AB + BC + CD + DE$ is greater than AE.

8. Let O be any point either inside or outside $\triangle ABC$.

Then $AO + BO$ is greater than AB,

$BO + CO$ is greater than BC,

$CO + AO$ is greater than CA; *I.* 20

$\therefore$ twice $(AO + BO + CO)$ is greater than $AB + BC + CA$;

$\therefore$ $AO + BO + CO$ is greater than half of $(AB + BC + CA)$.

When O is on one of the sides, as AC, the only modification to be made on the foregoing proof is to substitute $CO + AO = CA$, for $CO + AO$ is greater than CA.

9. Let ABC be the triangle.

Then $AB + BC$ is greater than CA; *I.* 20

$\therefore$ $AB + BC + CA$ is greater than twice CA;

$\therefore$ half of $(AB + BC + CA)$ is greater than CA.

Again, $AB + BC$ is greater than CA; *I.* 20

$\therefore$ twice $(AB + BC)$ is greater than $AB + BC + CA$;

$\therefore$ $AB + BC$ is greater than half of $(AB + BC + CA)$.

10. This is equivalent to the fourth deduction.

11. Let $ABCD$ be a quadrilateral whose diagonals AC, BD intersect at O.

By the preceding deduction, $AC + BD$ is greater than $AB + CD$,

and $AC + BD$ is greater than $BC + DA$;

$\therefore$ twice $(AC + BD)$ is greater than $AB + BC + CD + DA$.

Again, $AB + BC$ is greater than AC, *I.* 20
and $CD + DA$ is greater than AC; *I.* 20
$\therefore$ $AB + BC + CD + DA$ is greater than twice AC.
Hence also $AB + BC + CD + DA$ is greater than twice BD;
$\therefore$ twice $(AB + BC + CD + DA)$ is greater than twice $(AC + BD)$;
$\therefore$ $AB + BC + CD + DA$ is greater than $AC + BD$.

12. Let $ABCD$ be a quadrilateral whose diagonals are AC, BD.
Let P, any point different from the intersection of AC, BD, be joined to A, B, C, D.
Then $AP + CP$ is greater than AC,
and $BP + DP$ is greater than BD; *I.* 20
$\therefore$ $AP + BP + CP + DP$ is greater than $AC + BD$.

13. Let ABC be a triangle, AD the median from A.
Produce AD its own length to E, and join EC.

In $\triangle$s BDA, CDE, $\left\{\begin{array}{l} BD = CD \\ DA = DE \\ \angle BDA = \angle CDE; \end{array}\right.$ *I.* 15

$\therefore AB = EC$. *I.* 4
Hence $AB + AC = EC + AC$.
But $EC + AC$ is greater than AE or twice AD; *I.* 20
$\therefore$ $AB + AC$ is greater than twice AD.
Again, $AB - BD$ is less than AD, *I.* 20, *Cor.*
and $AC - CD$ is less than AD; *I.* 20, *Cor.*
$\therefore AB + AC - BD - CD$ is less than twice AD;
$\therefore$ $AB + AC - BC$ is less than twice AD.

14. Let ABC be a triangle, AH, BK, CL the three medians.
Then $AB + BC$ is greater than twice BK; $BC + CA$ is greater than twice CL; $CA + AB$ is greater than twice AH, by the preceding deduction;
$\therefore$ twice $(AB + BC + CA)$ is greater than twice $(AH + BK + CL)$;
$\therefore$ $AB + BC + CA$ is greater than $AH + BK + CL$.
Again, $AB + AC - BC$ is less than twice AH, by the preceding deduction,
$AB + BC - AC$ is less than twice BK,
$AC + BC - AB$ is less than twice CL;
$\therefore$ $AB + BC + CA$ is less than twice $(AH + BK + CL)$.

PROPOSITION 21.

1. Write out, word for word, the construction and proof given in the text; but wherever B occurs substitute C, and wherever C occurs substitute B.

2. Let AD produced meet BC at E.

   Then $\angle BDE$ is greater than $\angle BAD$, *I.* 16

   and $\angle CDE$ is greater than $\angle CAD$; *I.* 16

   $\therefore$ $\angle BDE + \angle CDE$ is greater than $\angle BAD + \angle CAD$;

   $\therefore$ $\angle BDC$ is greater than $\angle BAC$.

3. $AC + CB$ is greater than $AD + DB$; *I.* 21

   $\therefore AB + AC + CB$ is greater than $AB + AD + DB$.

4. Repeat, word for word, the preceding proof.

5. Let ABC be the triangle, O the point within it.

   Then $AO + BO$ is less than $BC + CA$, *I.* 21

   $BO + CO$ is less than $CA + AB$, *I.* 21

   $CO + AO$ is less than $AB + BC$; *I.* 21

   $\therefore$ twice $(AO + BO + CO)$ is less than twice $(AB + BC + CA)$;

   $\therefore AO + BO + CO$ is less than $AB + BC + CA$.

6. Let $\triangle ABC$ and quadrilateral $BDEC$ stand on the same base BC, and let the quadrilateral fall inside the triangle. Let DE produced meet AC at F.

   Then $DF + FC$ is greater than $DE + EC$; *I.* 21

   $\therefore BD + DF + FC$ is greater than $BD + DE + EC$.

   Again, $BA + AF$ is greater than $BD + DF$; *I.* 21

   $\therefore BA + AF + FC$ is greater than $BD + DF + FC$;

   $\therefore BA + AC$ is greater than $BD + DE + EC$;

   $\therefore BA + AC + BC$ is greater than $BD + DE + EC + BC$.

PROPOSITION 22.

1. If the circles intersect again at L, and LF, LG be joined, $\triangle$ LFG will fulfil the given conditions.

2. I. 1.

3, 4. When $A = B + C$, we have fig. 1; when A is greater than $B + C$, we have fig. 2.

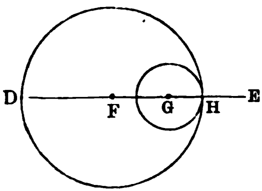


Fig. 1.

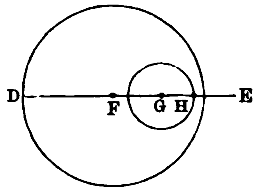


Fig. 2.

5. Let $ABCD$ be the given quadrilateral.
   Join AC;
   make $\triangle$ EFG having $EF = AB$, $FG = BC$, $GE = CA$; *I.* 22
   and on EG on the other side from F make $\triangle$ EGH having $GH = CD$, and $HE = AD$. *I.* 22
   $EFGH$ is the required quadrilateral.

6. Let $ABCDE$ be the given rectilineal figure.
   Join AD; make a quadrilateral $FGHK$ having $FG = AB$, $GH = BC$, $HK = CD$, and $KF = DA$, by the preceding deduction;
   on FK, on the other side from G and H, make $\triangle$ KLF having $KL = DE$, $LF = EA$. *I.* 22
   $FGHKL$ is the required rectilineal figure.

PROPOSITION 23.

1. Let AB be the given straight line, A the given point in it, and CDE the given angle.

   Produce ED to F; and at A make an angle $BAG = \angle$ CDF. *I.* 23

2. Let AB be the given straight line, A the given point in it, and CDE the given angle.
From D draw $DF \perp DE$; *I.* 11
and at A make an angle $BAG = \angle CDF$.

3. Let ABC be a triangle, having $\angle C = \angle A + \angle B$.
At C make $\angle BCD = \angle B$, and let CD meet AB at D. *I.* 23
Because $\angle BCD = \angle B$, $\therefore BD = CD$, *I.* 6
and $\triangle DBC$ is isosceles.
And because $\angle ACB = \angle A + \angle B$,
$\therefore \angle ACD = \angle A$, and $AD = CD$; *I.* 6
$\therefore \triangle DAC$ is isosceles.

4. At O without the angle AOB make $\angle BOE = \angle AOD$. *I.* 23
Then $\angle EOC = \angle DOC$, and $\angle DOE =$ twice $\angle DOC$.
Now $\angle DOE = \angle DOB + \angle BOE = \angle DOB + \angle DOA$;
$\therefore \angle DOB + \angle DOA =$ twice $\angle DOC$.

5. At O within the angle AOB make $\angle BOE = \angle AOD$. *I.* 23
Then $\angle EOC = \angle DOC$, and $\angle DOE =$ twice $\angle DOC$.
Now $\angle DOE = \angle DOB - \angle BOE = \angle DOB - \angle DOA$;
$\therefore \angle DOB - \angle DOA =$ twice $\angle DOC$.

6. Let AB be one of the equal sides.
At A make $\angle BAC =$ the given vertical angle; *I.* 23
cut off $AC = AB$, and join BC.

7. Let BC be the given base.
At B and C make $\angle CBA$ and $\angle BCA$ each $=$ the given base angle, and let BA, CA meet at A. *I.* 23

8. Make $BC =$ the given base; at C make $\angle BCA$ a right angle; cut off $CA =$ the given perpendicular; and join AB.

9. Make $BC =$ the given base; at C make $\angle BCA$ a right angle, and at B make $\angle CBA =$ the given acute angle; and let BA, CA meet at A.

10. Make $BC =$ the given base; at B and C make $\angle CBA$ and $\angle BCA$ respectively $=$ the given angles; and let BA, CA meet at A.

11. Make $AB =$ one of the given sides; at A make $\angle BAC =$ the given angle; cut off $AC =$ the other given side; and join BC.

12. Make $BC =$ the given base; at B make $\angle CBD =$ the given angle; cut off $BD =$ the sum of the other sides; join CD, and at C make $\angle DCA = \angle BDC$.
Let CA meet BD at A. ABC is the required triangle.
Because $\angle ACD = \angle ADC$, $\therefore AC = AD$; *I.* 6
$\therefore BA + AC = BD =$ given sum of sides.

13. Make $BC =$ the given base; at B make $\angle\ CBD =$ the given angle; cut off $BD =$ the difference of the other sides; join CD, and at C make $\angle\ DCA =$ the supplement of $\angle\ BDC$. Let CA meet BD produced at A.

ABC is the required triangle.

Because $\angle\ ACD = \angle\ ADC$, $\quad \therefore AC = AD$; *I.* 6

$\therefore BA - AC = BD =$ given difference of sides.

PROPOSITION 24.

1. In $\triangle$s AOB, BOC, $\left\{\begin{array}{l} AO = BO \\ OB = OC \\ \angle\ AOB \text{ is greater than } \angle\ BOC; \end{array}\right.$

$\therefore AB$ is greater than BC. *I.* 24

2. In $\triangle$s AOC, AOB, $\left\{\begin{array}{l} AO = AO \\ OC = OB \\ \angle\ AOC \text{ is greater than } \angle\ AOB; \end{array}\right.$

$\therefore AC$ is greater than AB. *I.* 24

In $\triangle$s AOC, BOC, $\left\{\begin{array}{l} AO = BO \\ OC = OC \\ \angle\ AOC \text{ is greater than } \angle\ BOC; \end{array}\right.$

$\therefore AC$ is greater than BC. *I.* 24

3. In $\triangle$s BCD, CBA, $\left\{\begin{array}{l} BC = CB \\ CD = BA \\ \angle\ BCD \text{ is greater than } \angle\ CBA; \end{array}\right.$

$\therefore BD$ is greater than CA. *I.* 24

4. Let $\angle\ BAD$ be greater than $\angle\ CAD$.

In $\triangle$s BAD, CAD, $\left\{\begin{array}{l} BA = CA \\ AD = AD \\ \angle\ BAD \text{ is greater than } \angle\ CAD; \end{array}\right.$

$\therefore BD$ is greater than CD. *I.* 24

5. Repeat, word for word, the construction and proof given in the text.

PROPOSITION 25.

1. In $\triangle$s AOB, BOC, $\left\{\begin{array}{l} AO = BO \\ OB = OC \\ AB \text{ is greater than } BC; \end{array}\right.$

$\therefore \angle AOB$ is greater than $\angle BOC$. *I.* 25

2. In $\triangle$s DCB, ABC, $\left\{\begin{array}{l} DC = AB \\ CB = BC \\ DB \text{ is greater than } AC; \end{array}\right.$

$\therefore \angle DCB$ is greater than $\angle ABC$. *I.* 25

3. By the third deduction from I. 24, BD is greater than AC.

In $\triangle$s DAB, ADC, $\left\{\begin{array}{l} DA = AD \\ AB = DC \\ DB \text{ is greater than } AC; \end{array}\right.$

$\therefore \angle DAB$ is greater than $\angle ADC$. *I.* 25

4. In $\triangle$s DAB, ADC, $\left\{\begin{array}{l} DA = AD \\ AB = DC \\ \angle DAB \text{ is greater than } \angle ADC; \end{array}\right.$

$\therefore DB$ is greater than AC. *I.* 24

In $\triangle$s DCB, ABC, $\left\{\begin{array}{l} DC = AB \\ CB = BC \\ DB \text{ is greater than } AC; \end{array}\right.$

$\therefore \angle DCB$ is greater than $\angle ABC$. *I.* 25

5. In $\triangle$s ADB, ADC, $\left\{\begin{array}{l} AD = AD \\ DB = DC \\ AB \text{ is less than } AC; \end{array}\right.$

$\therefore \angle ADB$ is less than $\angle ADC$. *I.* 25

But $\angle ADB + \angle ADC = 2$ rt. $\angle$s; *I.* 13

$\therefore \angle ADB$ is acute.

6. In $\triangle$s BAD, CAD, $\left\{\begin{array}{l} BA = CA \\ AD = AD \\ BD \text{ is greater than } CD; \end{array}\right.$

$\therefore \angle BAD$ is greater than $\angle CAD$. *I.* 25

7. By the fifth deduction from I. 25, $\angle ADB$ is acute, and

$\therefore \angle ADC$ is obtuse. *I.* 13

In $\triangle$s BDG, CDG, $\left\{\begin{array}{l} BD = CD \\ DG = DG \\ \angle BDG \text{ is less than } \angle CDG; \end{array}\right.$

$\therefore BG$ is less than CG. *I.* 24

PROPOSITION 26.

1. If $\triangle\, ABC$ be applied to $\triangle\, DEF$, so that B may fall on E, and so that BC may fall along EF, then C will coincide with F, because $BC = EF$;
   BA will fall along ED, because $\angle\, ABC = \angle\, DEF$;
   CA " FD, " $\angle\, ACB = \angle\, DFE$.
   Hence A will coincide with D, and $\triangle\, ABC$ with $\triangle\, DEF$;
   $\therefore AB = DE, AC = DF, \angle\, A = \angle\, D, \triangle\, ABC = \triangle\, DEF.$

2. Let ABC be an isosceles triangle, having $AB = AC$; and let AD bisect $\angle\, A$.

$$\text{In } \triangle\text{s } ABD, ACD, \left\{ \begin{array}{l} \angle\, ABD = \angle\, ACD \\ \angle\, BAD = \angle\, CAD \\ \quad AD = AD; \end{array} \right. \qquad \begin{array}{r} I.\,5 \\ Hyp. \\ \end{array}$$

   $\therefore BD = CD$, and $\angle\, ADB = \angle\, ADC$; *I.* 26
   $\therefore AD$ is $\perp BC$.

3. Let ABC be an isosceles triangle, having $AB = AC$; and let AD be $\perp BC$.

$$\text{In } \triangle\text{s } ABD, ACD, \left\{ \begin{array}{l} \angle\, ABD = \angle\, ACD \\ \angle\, ADB = \angle\, ADC \\ \quad AD = AD; \end{array} \right. \qquad I.\,5$$

   $\therefore BD = CD$, and $\angle\, BAD = \angle\, CAD$. *I.* 26

4. Let BAC be any angle, AD its bisector; from E, any point in AD, let EF, EG be respectively $\perp AB, AC$.

$$\text{In } \triangle\text{s } AFE, AGE, \left\{ \begin{array}{l} \angle\, EAF = \angle\, EAG \\ \angle\, EFA = \angle\, EGA \\ \quad AE = AE; \end{array} \right.$$

   $\therefore EF = EG$. *I.* 26

5. Let AB be a given straight line, CE, DE, which intersect at E, the two other straight lines.
   Bisect $\angle\, CED$ by EF, which meets AB or AB produced at G. *I.* 9
   Then, by the preceding deduction, G is equidistant from CE, DE.

6. Let A be a given point, B and C the two other given points.
   Join BC, and bisect it at D. *I.* 10
   AD is the required straight line.
   Draw BE, $CF \perp AD$ or AD produced.

In $\triangle$s BED, CFD, $\left\{\begin{array}{l} \angle BDE = \angle CDF \\ \angle BED = \angle CFD \\ BD = CD; \end{array}\right.$ *I.* 15

$\therefore BE = CF$. *I.* 26

7. Let A be the given point, BC, BD the two given intersecting straight lines.
   Bisect $\angle CBD$ by BE; *I.* 9
   from A draw $AF \perp BE$, meeting BC and BD in G, H.

   In $\triangle$s BFG, BFH, $\left\{\begin{array}{l} \angle GBF = \angle HBF \\ \angle GFB = \angle HFB \\ BF = BF; \end{array}\right.$

   $\therefore BG = BH$, and $\triangle BGH$ is isosceles. *I.* 26

PROPOSITION 27.

1. It was proved in I. 16, that $\angle EAB = \angle ECF$;
   $\therefore AB$ is $\parallel CF$. *I.* 27
2. In $\triangle$s AFE, CBE, $\left\{\begin{array}{l} AE = CE \\ EF = EB \\ \angle AEF = \angle CEB; \end{array}\right.$ *I.* 15
   $\therefore \angle AFE = \angle CBE$; *I.* 4
   $\therefore AF$ is $\parallel BC$. *I.* 27
3. Because $\angle AGE = \angle BGH$, and $\angle DHF = \angle CHG$; *I.* 15
   $\therefore \angle BGH = \angle CHG$;
   $\therefore AB$ is $\parallel CD$. *I.* 27
4. Because $\angle BGE = \angle AGH$, and $\angle CHF = \angle DHG$; *I.* 15
   $\therefore \angle AGH = \angle DHG$;
   $\therefore AB$ is $\parallel CD$. *I.* 27
5. Because $\angle CHF = \angle GHD$, *I.* 15
   $\therefore \angle AGE + \angle GHD = 2$ rt. $\angle$s.
   But $\angle AGE + \angle AGH = 2$ rt. $\angle$s; *I.* 13
   $\therefore \angle AGE + \angle GHD = \angle AGE + \angle AGH$;
   $\therefore \angle GHD = \angle AGH$;
   $\therefore AB$ is $\parallel CD$. *I.* 27
6. Because $\angle DHF = \angle GHC$, *I.* 15
   $\therefore \angle BGE + \angle GHC = 2$ rt. $\angle$s.
   But $\angle BGE + \angle BGH = 2$ rt. $\angle$s; *I.* 13
   $\therefore \angle BGE + \angle GHC = \angle BGE + \angle BGH$;
   $\therefore \angle GHC = \angle BGH$; *I.* 8
   $\therefore AB$ is $\parallel CD$. *I.* 27

7, 8. Let $ABCD$ be a square or a rhombus.

Join BD.

In $\triangle$s ADB, CBD, $\left\{\begin{array}{l} AD = CB \\ DB = BD \\ AB = CD; \end{array}\right.$

$\therefore \angle ADB = \angle CBD;$ *I.* 8

$\therefore AD$ is $\parallel BC$. *I.* 27

Hence also AB is $\parallel CD$.

9. Let $ABCD$ be a quadrilateral whose diagonals AC, BD bisect one another at E.

In $\triangle$s AEB, CED, $\left\{\begin{array}{l} AE = CE \\ EB = ED \\ \angle AEB = \angle CED; \end{array}\right.$ *I.* 15

$\therefore \angle ABE = \angle CDE;$ *I.* 4

$\therefore AB$ is $\parallel CD$. *I.* 27

Hence also AD is $\parallel BC$, and $ABCD$ is a $\parallel^{m}$.

PROPOSITION 28.

1. Because $\angle DHG + \angle DHF = 2$ rt. $\angle$s, *I.* 13

$\therefore \angle BGE + \angle DHF = \angle DHG + \angle DHF;$

$\therefore \angle BGE = \angle DHG;$

$\therefore AB$ is $\parallel CD$. *I.* 28

2. Because $\angle CHG + \angle CHF = 2$ rt. $\angle$s, *I.* 13

$\therefore \angle AGE + \angle CHF = \angle CHG + \angle CHF;$

$\therefore \angle AGE = \angle CHG;$

$\therefore AB$ is $\parallel CD$. *I.* 28

3. Because $\angle DHF = \angle CHG$, *I.* 15

$\therefore \angle AGE = \angle CHG;$

$\therefore AB$ is $\parallel CD$. *I.* 28

4. Because $\angle CHF = \angle DHG$, *I.* 15

$\therefore \angle BGE = \angle DHG;$

$\therefore AB$ is $\parallel CD$. *I.* 28

5. Let $ABCD$ be a square.

Because $\angle ABC + \angle BCD = 2$ rt. $\angle$s,

$\therefore AB$ is $\parallel CD$. *I.* 28

Hence also AD is $\parallel BC$.

6. Because $\angle A + \angle B = 2$ rt. $\angle$s,
$\therefore AD$ is $\parallel BC$. *I.* 28
Because $\angle B + \angle C = 2$ rt. $\angle$s,
$\therefore AB$ is $\parallel CD$. *I.* 28
Hence $ABCD$ is a $\parallel^m$.

PROPOSITION 29.

1. Because $\angle BGH = \angle GHC$, *I.* 29
and $\angle AGE = \angle BGH$, and $\angle DHF = \angle GHC$; *I.* 15
$\therefore \angle AGE = \angle DHF$.
Again, because $\angle BGE = \angle DHG$, *I.* 29
$\therefore \angle BGE + \angle DHF = \angle DHG + \angle DHF$,
$= 2$ rt. $\angle$s. *I.* 13

2. Because the two interior angles which the straight line makes with the parallels are $= 2$ rt. $\angle$s; *I.* 29
$\therefore$ if one of them be a right angle, so must the other.

3. Let ABC be an isosceles triangle, having $AB = AC$; and let DE drawn $\parallel BC$ meet AB, AC, or AB, AC produced, either below the base or through the vertex, at D, E.
Because ADB (or ABD, DAB) cuts the parallels DE, BC,
$\therefore \angle ADE = \angle ABC$. *I.* 29
Because AEC (or ACE, EAC) cuts the parallels DE, BC,
$\therefore \angle AED = \angle ACB$. *I.* 29
But $\angle ABC = \angle ACB$; *I.* 5
$\therefore \angle ADE = \angle AED$, and $\triangle ADE$ is isosceles. *I.* 6

4. Let AB, BC be respectively $\parallel DE$, EF, and drawn both in similar or both in opposite directions.
Produce DE, if necessary, to meet BC at G.
Then $\angle ABC = \angle DGC$, *I.* 29
$= \angle DEF$. *I.* 29
When AB, DE are drawn in similar directions,
but BC, EF are drawn in opposite directions,
$\angle ABC$ is supplementary to $\angle DEF$.
For, as before, $\angle ABC = \angle DGC$, *I.* 29
$= \angle GEF$; *I.* 29
and $\angle GEF$ is supplementary to $\angle DEF$. *I.* 13

5. Not necessarily.
In the figure on p. 240 of *Elements of Euclid*,

$\angle CAB = \angle EDA$, and CA is $\parallel ED$,
but AB is not $\parallel DA$.

6, 7. Let L be the middle point of GH (fig. to I. 28), and let MN be drawn through L, and terminated by the parallels.

$$\text{In } \triangle\text{s } GLM, HLN, \begin{cases} \angle MGL = \angle NHL & I.\ 29 \\ \angle GML = \angle HNL & I.\ 29 \\ LG = LH; \end{cases}$$

$\therefore LM = LN$, and $GM = HN$. *I.* 26

8. Let ABC be an isosceles triangle, having $AB = AC$; and let AE be parallel to BC, and BA be produced to D.

Then $\angle DAE = \angle ABC$, and $\angle EAC = \angle ACB$. *I.* 29
But $\angle ABC = \angle ACB$; *I.* 5
$\therefore \angle DAE = \angle EAC$.

9. Let ABC be a triangle, having BA produced to D; and let AE, which bisects $\angle CAD$, be $\parallel BC$.

Then $\angle DAE = \angle ABC$, and $\angle EAC = \angle ACB$. *I.* 29
But $\angle DAE = \angle EAC$; *Hyp.*
$\therefore \angle ABC = \angle ACB$, and $\triangle ABC$ is isosceles. *I.* 6

10. Let $ABCD$ be a $\parallel^{m}$, and let its diagonals AC, BD intersect at E.

$$\text{In } \triangle\text{s } ADB, CBD, \begin{cases} \angle ADB = \angle CBD & I.\ 29 \\ \angle ABD = \angle CDB & I.\ 29 \\ DB = BD; \end{cases}$$

$\therefore AD = CB$. *I.* 26

$$\text{In } \triangle\text{s } AED, CEB, \begin{cases} \angle ADE = \angle CBE & I.\ 29 \\ \angle DAE = \angle BCE & I.\ 29 \\ AD = CB; \end{cases}$$

$\therefore AE = CE$, $DE = BE$. *I.* 26

11. Because $\angle CDE = \angle DEF$, each being right,
$\therefore CD$ is $\parallel EF$; *I.* 27
$\therefore \angle DCF = \angle EFC$, *I.* 29
$= \angle ECF$. *I.* 5

PROPOSITION 30.

1. Let $ABCD$, $EBCF$ be $\parallel^{ms}$, having a common base BC.

Since AD and EF are both $\parallel BC$, *I. Def.* 33
$\therefore AD$ is $\parallel EF$. *I.* 30

2. Since AB is $\parallel EF$, $\therefore \angle AGH = \angle HKF$; *I.* 29
and since CD is $\parallel EF$, $\therefore \angle GHD = \angle HKF$; *I.* 29
$\therefore \angle AGH = \angle GHD$, and AB is $\parallel CD$. *I.* 27

PROPOSITION 31.

1. From A draw $AD \perp BC$, *I.* 12
and at A draw $AE \perp AD$, and produce EA to F. *I.* 11
Because $\angle EAD + \angle ADB = 2$ rt. $\angle$s,
$\therefore$ EF is $\parallel BC$. *I.* 28

2. Let A be the given point, BC the given straight line, and D the given angle.
Through A draw $EF \parallel BC$; *I.* 31
at A make $\angle EAG = \angle D$, *I.* 23
and let AG meet BC at G.
Because EF is $\parallel BC$, $\therefore$ $\angle AGC = \angle EAG$, *I.* 29
$= \angle D$. *Const.*

3. Let A be the given point, BC, BD the given intersecting straight lines.
In BC take any point E, and from BD cut off $BF = BE$; *I.* 3
join EF, and through A draw $AGH \parallel EF$, *I.* 31
meeting BC at G, and BD at H.
Because $BE = BF$, $\therefore$ $\angle BEF = \angle BFE$. *I.* 5
But $\angle BEF = \angle BGH$, and $\angle BFE = \angle BHG$; *I.* 29
$\therefore$ $\angle BGH = \angle BHG$, and $\triangle BGH$ is isosceles. *I.* 6

4. Let AB, CD be the parallels, E the given straight line, and P the given point, not situated in AB or CD.
Take any point F in AB, and with F as centre and a radius $= E$, describe a circle cutting CD in G and H; join FG, FH, and through P draw $PMN \parallel FG$, and $PQR \parallel FH$, the points M, Q being on AB, and N, R on CD.
Then $FGNM$ and $FHRQ$ are $\parallel^{ms}$; *I. Def.* 33
and FG may be proved $= MN$, and $FH = QR$, as in the first part of the tenth deduction from *I.* 29.
But FG and FH are each $= E$;
$\therefore$ MN and QR are each $= E$.
If the circle described with F as centre and a radius $= E$ meets CD in only one point, there can be only one solution; if it does not meet CD at all, the problem is impossible. The circle will not meet CD at all when E is less than the perpendicular drawn from F to CD.

PROPOSITION 32.

1. The vertical angle of the isosceles triangle is right, since if either of the base angles were right, the other also would be right. *I.* 5
   Hence the two base angles must be equal to a right angle; *I.* 32
   and since they are equal, each must be half a right angle.
2. Let A and D denote the vertical angles of two isosceles triangles ABC, DEF.
   Since $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$, *I.* 32
   and $\angle A = \angle D$;
   $\therefore \angle B + \angle C = \angle E + \angle F$.
   But $\angle B + \angle C =$ twice $\angle B$,
   and $\angle E + \angle F =$ twice $\angle E$; *I.* 5
   $\therefore$ twice $\angle B =$ twice $\angle E$, and $\angle B = \angle E$.
   Hence also $\angle C = \angle F$.
3. Let $\angle A$ of $\triangle ABC$ be $= \angle B + \angle C$.
   Then $\angle A + \angle A = \angle A + \angle B + \angle C$;
   $\therefore$ twice $\angle A = 2$ rt. $\angle$s; *I.* 32
   $\therefore \angle A =$ a rt. $\angle$.
4. Let $\angle A$ of $\triangle ABC$ be greater than $\angle B + \angle C$.
   Then $\angle A + \angle A$ is greater than $\angle A + \angle B + \angle C$;
   $\therefore$ twice $\angle A$ is greater than 2 rt. $\angle$s; *I.* 32
   $\therefore \angle A$ is greater than a rt. $\angle$.
5. Let $\angle A$ of $\triangle ABC$ be less than $\angle B + \angle C$.
   Then $\angle A + \angle A$ is less than $\angle A + \angle B + \angle C$;
   $\therefore$ twice $\angle A$ is less than 2 rt. $\angle$s; *I.* 32
   $\therefore \angle A$ is less than a rt. $\angle$.
6. Since a right-angled triangle has one of its angles equal to the sum of the other two, this deduction is the same as the third from I. 23.
7. In the proof given of the third deduction from I. 23, it is shown that AD, BD, CD are all equal.
8. Let BC be the given straight line.
   On BC as base describe any isosceles $\triangle DBC$;
   produce BD its own length to A, and join AC.
   Then $\angle DCA = \angle A$, and $\angle DCB = \angle B$; *I.* 5
   $\therefore \angle ACB = \angle A + \angle B$;
   $\therefore \angle ACB$ is right, by the third deduction from I. 32.

9. Since all the angles of an equilateral triangle are equal,
and since they are together $= 2$ rt. $\angle$s; *I.* 32
$\therefore$ each angle $=$ one-third of 2 rt. $\angle$s,
$=$ two-thirds of a rt. $\angle$.

10. Let $\angle$ *ABC* be right.
On *BC* describe an equilateral $\triangle$ *DBC*. *I.* 1
Then by the preceding deduction $\angle$ *DBC* $=$ two-thirds of a rt. $\angle$;
$\therefore$ $\angle$ *ABD* $=$ one-third of $\angle$ *ABC*.
Bisect $\angle$ *DBC* to obtain the other thirds of $\angle$ *ABC*.

11. Since $\angle B = \angle DAB$, and $\angle C = \angle EAC$, *I.* 29
$\therefore \angle B + \angle C + \angle BAC = \angle DAB + \angle EAC + \angle BAC$,
$= 2$ rt. $\angle$s. *I.* 13

12. Let the vertical angle be two-thirds of a right angle, or one-third of two right angles;
then the sum of the base angles is two-thirds of two right angles.
Now since the base angles are equal, each is one-third of two right angles, and the triangle is equilateral.
Again, let one of the base angles be two-thirds of a right angle, or one-third of two right angles;
then the other base angle is also one-third of two right angles;
and consequently the vertical angle must be one-third of two right angles, and the triangle is equilateral.

13. The exterior vertical angle is equal to the sum of the base angles. *I.* 32
But the base angles are equal; *I.* 5
$\therefore$ each of them is half of the exterior vertical angle.

14. Let *ABC* be an isosceles triangle, having $AB = AC$; let *BA* be produced to *D*, and let $\angle$ *CAD* be bisected by *AE*.
By the preceding deduction, $\angle EAC = \angle ACB$;
$\therefore$ *AE* is $\parallel$ *BC*. *I.* 27

15. Each angle of an equilateral triangle $= \frac{1}{3}$ of 2 rt. $\angle$s;
$\therefore$ six such angles $= 4$ rt. $\angle$s.
Each angle of a square $=$ a rt. $\angle$;
$\therefore$ four such angles $= 4$ rt. $\angle$s.
All the angles of a regular hexagon $= 8$ rt. $\angle$s; *I.* 32, *Cor.* 3
and since all the angles are equal,
$\therefore$ three such angles $= 4$ rt. $\angle$s.

16. A right angle can be divided into 4, 8, 16, 32, 64, &c. equal parts, and also into 6, 12, 24, 48, 96, &c. equal parts.
See *Elements of Euclid*, p. 239.

17. This deduction is part of VI. 8. See *Elements of Euclid*, p. 302.

18. Let ABC be a triangle, right-angled at C, and let D be the middle point of the hypotenuse AB.

INDIRECTLY.

If DC be not $= DA$ or DB, suppose it greater.
Then $\angle A$ is greater than $\angle DCA$, *I.* 18
and $\angle B$ is greater than $\angle DCB$; *I.* 18
$\therefore$ $\angle A + \angle B$ is greater than $\angle ACB$;
$\therefore$ $\angle A + \angle B + \angle ACB$ is greater than 2 rt. $\angle$s,
which is impossible. *I.* 32
Similarly, DC is not less than DA or DB.

DIRECTLY.

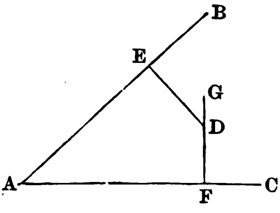
Produce CD its own length to E, and join EB.

In $\triangle$s ACD, BED, $\begin{cases} AD = BD & \textit{Hyp.} \\ DC = DE & \textit{Const.} \\ \angle ADC = \angle BDE; & \textit{I. } 15 \end{cases}$

$\therefore AC = BE$, $\angle ACD = \angle BED$. *I.* 4
Because $\angle ACD = \angle BED$, $\therefore AC$ is $\parallel BE$; *I.* 27
$\therefore \angle ACB + \angle EBC = 2$ rt. $\angle$s; *I.* 29
$\therefore \angle EBC$ is right.

In $\triangle$s ACB, EBC, $\begin{cases} AC = EB \\ CB = BC \\ \angle ACB = \angle EBC; \end{cases}$

$\therefore AB = EC$; *I.* 4
$\therefore$ half of AB = half of EC,
or $AD = CD$.



19. Let DE, DF be respectively $\perp AB$, AC.
Then $AEDF$ is a quadrilateral;
$\therefore$ its four interior angles $= 4$ rt. $\angle$s.
But $\angle AED + \angle AFD = 2$ rt. $\angle$s;
$\therefore \angle A + \angle EDF = 2$ rt. $\angle$s;
$\therefore \angle A$ is supplementary to $\angle EDF$.

Again $\angle$ *EDG* is supplementary to $\angle$ *EDF*; *I.* 13
$\therefore$ $\angle A = \angle EDG$.

20. Let O be a point inside *ABCDE* (fig. to I. 32, Cor. 3); join O to A, B, C, D, E.
Then there are as many triangles as the figure has sides;
$\therefore$ all the angles of these triangles are = twice as many right angles as the figure has sides.
But all the angles of these triangles make up the interior angles of the figure together with the angles at O;
and the angles at O are = 4 rt. $\angle$s;
$\therefore$ the interior angles of the figure, together with 4 rt. $\angle$s, are = twice as many right angles as the figure has sides.
If the figure has n sides,
then its interior angles + 4 rt. $\angle$s = $2n$ rt. $\angle$s;
therefore its interior angles = $(2n - 4)$ rt. $\angle$s.

PROPOSITION 33.

1. The proposition itself.

2. If one pair of opposite sides be equal and parallel, the other pair will be equal and parallel; *I.* 33
$\therefore$ the quadrilateral is a $\parallel^{m}$.

3. They bisect each other.

PROPOSITION 34.

1. Let *ABDC* be a $\parallel^{m}$ (fig. to I. 34), and let $AB = AC$.
Since $AB = CD$, and $AC = BD$, *I.* 34
$\therefore$ $BD = CD$, and all the sides are equal.

2. Let *ABDC* be a $\parallel^{m}$, and let $\angle A = \angle B$.
Since $\angle A + \angle B = 2$ rt. $\angle$s, *I.* 29
$\therefore$ $\angle A$ and $\angle B$ are each right;
$\therefore$ $\angle D$ and $\angle C$ are each right. *I.* 34

3. Let *ABDC* be a $\parallel^{m}$, and let $\angle A$ be right.
Since $\angle A + \angle B = 2$ rt. $\angle$s, *I.* 29
$\therefore$ $\angle B$ is right;
$\therefore$ $\angle C$ and $\angle D$ are each right. *I.* 34

4. Let $ABCD$, $EFGH$ be two $\|^{ms}$, having $\angle A = \angle E$.

Since $\angle A + \angle B = 2$ rt. $\angle$s, *I.* 29

and $\angle E + \angle F = 2$ rt. $\angle$s; *I.* 29

$\therefore \angle A + \angle B = \angle E + \angle F;$

$\therefore \angle B = \angle F.$

Now, $\angle A = \angle C$, and $\angle E = \angle G$; *I.* 34

$\therefore \angle C = \angle G.$

And $\angle B = \angle D$, and $\angle F = \angle H$; *I.* 34

$\therefore \angle D = \angle H.$

5. Let $ABDC$ be a quadrilateral (fig. to I. 34), having $AB = CD$, and $AC = BD$.

Join BC.

In $\triangle$s ABC, DCB, $\left\{\begin{array}{l} AB = DC \\ BC = CB \\ CA = BD; \end{array}\right.$

$\therefore \angle ABC = \angle DCB$, and $\angle ACB = \angle DBC$; *I.* 8

$\therefore AB$ is $\|$ CD, and AC is $\|$ BD; *I.* 27

$\therefore ABDC$ is a $\|^m$.

6. Let $ABCD$ be a quadrilateral, having $\angle A = \angle C$, and $\angle B = \angle D$.

Then $\angle A + \angle B = \angle C + \angle D.$

But $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s; *I.* 32, *Cor.* 2

$\therefore \angle A + \angle B = 2$ rt. $\angle$s;

$\therefore AD$ is $\|$ BC. *I.* 28

Similarly, AB is $\|$ DC, and the quadrilateral is a $\|^m$.

7. Let $ABCD$ be a $\|^m$, and let $AC = BD$.

In $\triangle$s ABC, DCB, $\left\{\begin{array}{l} AB = DC \\ BC = CB \\ AC = DB; \end{array}\right.$

$\therefore \angle ABC = \angle DCB.$ *I.* 8

But $\angle ABC + \angle DCB = 2$ rt. $\angle$s; *I.* 29

$\therefore \angle ABC$ and $\angle DCB$ are each right;

$\therefore \angle ADC$ and $\angle DAB$ are each right; *I.* 34

and the $\|^m$ is a rectangle.

8. Let $ABCD$ be a $\|^m$, and let AC, BD bisect the angles through which they pass.

Then $\angle ADB = \angle CBD.$ *I.* 29

But $\angle CBD = \angle ABD;$

$\therefore \angle ADB = \angle ABD$, and $AB = AD$. *I.* 6

Hence by the first deduction from I. 34, the $\|^m$ is a rhombus.

9. Let $ABCD$ be a $||^m$, and let AC, BD cut each other perpendicularly at E.

By the tenth deduction from I. 29, $AE = CE$, $BE = DE$.

$$\text{In } \triangle\text{s } AEB, AED, \left\{\begin{array}{c} BE = DE \\ EA = EA \\ \angle BEA = \angle DEA; \end{array}\right.$$

$\therefore AB = AD$. *I.* 4

Hence by the first deduction from I. 34, the $||^m$ is a rhombus.

10. Since the diagonals are equal, the $||^m$ is a rectangle; and since they cut each other perpendicularly, the $||^m$ is a rhombus;

$\therefore$ the $||^m$ is a square.

11. Let BC be the given straight line (fig. to I. 34).

Through B and C draw the parallels BA, CD;

and through B and C draw the parallels BD, CA.

Join AD.

The figure $ABDC$ is a $||^m$;

$\therefore$ by the tenth deduction from I. 29, AD bisects BC.

12. Let $ABCD$ be a $||^m$, and let AC, BD intersect at E; through E let there be drawn any straight line FEG, meeting AD and BC in F and G.

$$\text{In } \triangle\text{s } AFE, CGE, \left\{\begin{array}{ll} \angle AFE = \angle CGE & I.\ 29 \\ \angle FAE = \angle GCE & I.\ 29 \\ AE = CE; & \end{array}\right.$$

$\therefore FE = GE$, and $\triangle AFE = \triangle CGE$. *I.* 26

From $\triangle ABC$, which is half of the $||^m$ $ABCD$, *I.* 34

take away $\triangle CGE$, and add on $\triangle AFE$;

then the figure $ABGF$ is half of the $||^m$ $ABCD$.

13. Let $ABCD$ be a $||^m$, P any point.

Join AC, BD intersecting at E; join PE, and produce it to meet a pair of opposite sides of the $||^m$.

The proof follows from the preceding deduction.

14. Let ABC be a triangle, L and K the middle points of AB, AC.

Produce LK its own length to M, and join CM.

$$\text{In } \triangle\text{s } AKL, CKM, \left\{\begin{array}{c} AK = CK \\ KL = KM \\ \angle AKL = \angle CKM; \end{array}\right. \qquad I.\ 15$$

$\therefore AL = CM$, and $\angle KAL = \angle KCM$. *I.* 4

Now $AL = BL$; $\therefore BL = CM$.

And since $\angle\, KAL = \angle\, KCM$,
$\therefore$ BL is $\parallel$ CM; I. 27
$\therefore$ $LBCM$ is a $\parallel^{m}$. I. 33
Hence LM is $\parallel$ BC, and $= BC$;
$\therefore$ LK is $\parallel$ BC, and $=$ half of BC.

15. Let ABC be a triangle, H, K, L the middle points of BC, CA, AB.

From the preceding deduction it follows that $ALHK$, $BHKL$, $CKLH$ are $\parallel^{ms}$, of which KL, LH, HK are diagonals;
$\therefore$ each of the triangles ALK, BHL, $CKH = \triangle\, HKL$.

16. Let H, K, L be the middle points of the sides of a triangle; to construct the triangle.

Join HK, KL, LH; through H draw $BC \parallel KL$, through K draw $CA \parallel LH$, and through L draw $AB \parallel HK$.
ABC is the required triangle.
For $ALHK$, $BHKL$ are $\parallel^{ms}$;
$\therefore$ $AL = HK$, and $BL = HK$; I. 34
$\therefore$ $AL = BL$, or L is the middle point of AB.
Similarly, H and K are the middle points of BC, CA.

PROPOSITION 35.

1. The $\parallel^{ms}$ then become $ABCD$, $DBCF$;
and they are equal, since each is double of $\triangle\, DBC$. I. 34

2. Let $ABCD$, $EBCF$ be two equal $\parallel^{ms}$ on the same base BC, and on the same side of it.
If AD, or AD produced, be not in the same straight line with EF, let it cut BE and CF, or these straight lines produced, at G and H.

Then $GBCH$ is a $\parallel^{m}$, and $= ABCD$. I. 35
But $ABCD = EBCF$;
$\therefore$ $GBCH = EBCF$,
which is impossible, since the one is a part of the other.
Hence AD and EF must be in the same straight line;
that is, $ABCD$ and $EBCF$ are between the same parallels.

3. Let the figure be lettered as in the fifteenth deduction from I. 34.

Then $\|^{m}$ $ALHK$ and $\|^{m}$ $BHKL$ are on the same base HK, and between the same parallels AB, HK;

$\therefore$ $\|^{m}$ $ALHK = \|^{m}$ $BHKL$.

Similarly, $\|^{m}$ $BHKL = \|^{m}$ $CKLH$.

4. Let $ABCD$ be the given $\|^{m}$ (fig. to I. 35).

With B as centre, and radius BC, describe a circle cutting AD or AD produced at E; join BE, and through C draw $CF \parallel BE$, meeting AD or AD produced at F.

$EBCF$ is the required rhombus.

For $EBCF$ is a $\|^{m}$, and $=$ $\|^{m}$ $ABCD$; *I.* 35

and $EBCF$ is a rhombus, by the first deduction from I. 34.

If the circle, with B as centre, and BC as radius, do not cut AD or AD produced, construct the rhombus on the base AB.

5. Because $AD = BC$, and $EF = BC$, *I.* 34

$\therefore \quad AD = EF$.

Add to, or take away from, these equals DE;

then the whole or remainder EA = whole or remainder FD.

In $\triangle$s EAB, FDC, $\left\{\begin{array}{l} EA = FD \\ AB = DC \quad\quad I.\ 34 \\ \angle EAB = \angle FDC; \quad\quad I.\ 29 \end{array}\right.$

$\therefore$ $\triangle EAB = \triangle FDC$. *I.* 4

OR,

As before, prove $EA = FD$.

In $\triangle$s EAB, FDC, $\left\{\begin{array}{l} EA = FD \\ AB = DC \quad\quad I.\ 34 \\ BE = CF; \quad\quad I.\ 34 \end{array}\right.$

$\therefore$ $\triangle EAB = \triangle FDC$. *I.* 8

PROPOSITION 36.

1. Because $BC = FG$, and $BC = AD$, *Hyp.*, *I.* 34

$\therefore \quad AD = FG$.

And because AD is $\parallel FG$,

$\therefore$ $AFGD$ is a $\|^{m}$.

Now $\|^{m}$ $ABCD = \|^{m}$ $AFGD$, *I.* 35
and $\|^{m}$ $EFGH = \|^{m}$ $AFGD$; *I.* 35
$\therefore$ $\|^{m}$ $ABCD = \|^{m}$ $EFGH$.

2. Let $ABCD$ be a $\|^{m}$.
Bisect BC at E, and through E draw $EF \parallel AB$, meeting AD in F.
Then $ABEF$, $FECD$ are $\|^{ms}$,
and they are equal, because they are on equal bases BE, EC, and between the same parallels AD, BC; *I.* 36
$\therefore$ $\|^{m}$ $ABCD$ is divided into two equal $\|^{ms}$.

3. In two ways. AB may be bisected at E, and EF drawn $\parallel BC$.

4. Let $ABCD$, $EFGH$ be two $\|^{ms}$ between the same parallels AH, BG, but let the base BC be greater than the base FG.
From BC cut off $BK = FG$, and through K draw $KL \parallel AB$, meeting AB at L.
Then $\|^{m}$ $ABKL = \|^{m}$ $EFGH$. *I.* 36
But $\|^{m}$ $ABCD$ is greater than $\|^{m}$ $ABKL$;
$\therefore$ $\|^{m}$ $ABCD$ is greater than $\|^{m}$ $EFGH$.

5. Of two $\|^{ms}$ which are between the same parallels, that which is the greater stands on the greater base.
Let $ABCD$, $EFGH$ be two $\|^{ms}$ between the same parallels AH, BG, but let $\|^{m}$ $ABCD$ be greater than $\|^{m}$ $EFGH$.
The base BC cannot be $=$ the base FG,
for then $\|^{m}$ $ABCD$ would be $= \|^{m}$ $EFGH$. *I.* 36
The base FG cannot be greater than the base BC,
for then $\|^{m}$ $EFGH$ would be greater than $\|^{m}$ $ABCD$, by the preceding deduction;
$\therefore$ the base BC is greater than the base FG.

6. Let $ABCD$, $EFGH$ be equal $\|^{ms}$ between the same parallels AH, BG.
The base BC cannot be greater than the base FG,
for then $\|^{m}$ $ABCD$ would be greater than $\|^{m}$ $EFGH$ by the fourth deduction.
For a similar reason the base FG cannot be greater than the base BC;
$\therefore$ $BC = FG$.

PROPOSITION 37.

1. Because $\triangle$s DBC, EBC are on the same base BC and between the same parallels DE, BC, $\therefore$ they are equal. *I.* 37
Because $\triangle$s BDE, CED are on the same base DE and between the same parallels BC, DE, $\therefore$ they are equal. *I.* 37
Because $\triangle\ BDE = \triangle\ CED$,
$\therefore\ \triangle\ BDE + \triangle\ ADE = \triangle\ CED + \triangle\ ADE;$
$\therefore\qquad \triangle\ ABE = \triangle\ ACD.$
2. Because $\qquad \triangle\ ADC = \triangle\ BDC.$ *I.* 37
$\therefore\ \triangle\ ADC - \triangle\ ODC = \triangle\ BDC - \triangle\ ODC;$
$\therefore\qquad \triangle\ AOD = \triangle\ BOC.$
3. If AD were $= BC$, $ABCD$ would be a $\parallel^m$;
and $\triangle$s ABC, DBC would be equal, since each is half of the $\parallel^m$ $ABCD$.
4. Let $ABCD$ be the given quadrilateral.
Join AC, and through D draw $DE \parallel AC$ to meet BC produced at E; join AE.
Then $\qquad \triangle\ ACE = \triangle\ ACD;$ *I.* 37
$\therefore\ \triangle\ ACE + \triangle\ ABC = \triangle\ ACD + \triangle\ ABC;$
$\therefore\qquad \triangle\ ABE =$ quadrilateral $ABCD$.
5. Join DC, and through A draw $AE \parallel DC$ to meet BC produced at E; join DE.
Then $\qquad \triangle\ DCE = \triangle\ ADC;$ *I.* 37
$\therefore\ \triangle\ DCE + \triangle\ DBC = \triangle\ ADC + \triangle\ DBC;$
$\therefore\qquad \triangle\ DBE = \triangle\ ABC.$

PROPOSITION 38.

1. Let ABC, DEF be two triangles between the same parallels AD, BF, but let the base BC be greater than the base EF.
From BC cut off $BK = EF$, and join AK.
Then $\triangle\ ABK = \triangle\ DEF.$ *I.* 38
But $\triangle\ ABC$ is greater than $\triangle\ ABK$;
$\therefore\ \triangle\ ABC$ is greater than $\triangle\ DEF$.
2. Of two triangles which are between the same parallels, that which is the greater stands on the greater base.
Let ABC, DEF be two triangles between the same parallels AD, BF, but let $\triangle\ ABC$ be greater than $\triangle\ DEF$.

The base BC cannot be = the base EF,
for then $\triangle\ ABC$ would be $= \triangle\ DEF$. *I.* 38
The base EF cannot be greater than the base BC,
for then $\triangle\ DEF$ would be greater than $\triangle\ ABC$, by the preceding deduction;
$\therefore$ the base BC is greater than the base EF.

3. Let ABC, DEF be two triangles between the same parallels AD, BF, but let the base BC be double of the base EF.
Bisect BC at G, and join AG.
Then $\triangle\ ABC$ is double of $\triangle\ ABG$. *I.* 38, *Cor.*
But $\triangle\ ABG = \triangle\ DEF$; *I.* 38
$\therefore\ \triangle\ ABC$ is double of $\triangle\ DEF$.

4. Let $ABCD$ be a $\parallel^m$, and let its diagonals AC, BD intersect at E.
By the tenth deduction from I. 29, the diagonals of a $\parallel^m$ bisect each other; $\therefore\ AE = EC$, and $BE = ED$.
Because $AE = EC$, $\therefore\ \triangle\ AEB = \triangle\ CEB$. *I.* 38
Because $BE = ED$, $\therefore\ \triangle\ CEB = \triangle\ CED$. *I.* 38
Similarly, $\triangle\ CED = \triangle\ AED$; *I.* 38
$\therefore$ the four $\triangle$s AEB, AED, CEB, CED are all equal.

5. Let $ABCD$ be a quadrilateral, and let the diagonal AC bisect BD at E.
Because $BE = ED$,
$\therefore\ \triangle\ AEB = \triangle\ AED$, and $\triangle\ CEB = \triangle\ CED$; *I.* 38
$\therefore\ \triangle\ AEB + \triangle\ CEB = \triangle\ AED + \triangle\ CED$;
$\therefore\ \triangle\ ABC = \triangle\ ADC$.

6. Because $ABCD$ is a $\parallel^m$, $\therefore\ AD = BC$. *I.* 34
Because $\triangle$s AFD, BEC are on equal bases AD, BC, and between the same parallels AE, BF,
$\therefore\ \triangle\ AFD = \triangle\ BEC$. *I.* 38

7. Because $AK = CK$.
$\therefore\ \triangle\ AKB = \triangle\ CKB$, and $\triangle\ AKG = \triangle\ CKG$; *I.* 38
$\therefore\ \triangle\ AKB - \triangle\ AKG = \triangle\ CKB - \triangle\ CKG$;
$\therefore\ \triangle\ AGB = \triangle\ BGC$.
Hence also, $\triangle\ AGC = \triangle\ BGC$;
$\therefore\ \triangle\ BGC = \triangle\ AGC = \triangle\ AGB$.

8. Join AC cutting BD at E. [Suppose P to be situated in ED. The demonstration is easily modified to suit any other position of P.]
Because $AE = EC$, by the tenth deduction from I. 29,
$\therefore\ \triangle\ AEB = \triangle\ CEB$, and $\triangle\ AEP = \triangle\ CEP$; *I.* 38

$\therefore \triangle AEB + \triangle AEP = \triangle CEB + \triangle CEP$;

$\therefore \triangle PAB = \triangle PCB$.

Because $AE = EC$,

$\therefore \triangle AED = \triangle CED$, and $\triangle AEP = \triangle CEP$; *I.* 38

$\therefore \triangle AED - \triangle AEP = \triangle CED - \triangle CEP$;

$\therefore \triangle PAD = \triangle PCD$.

9. Let ABC be a triangle, D any point in AB.

Bisect BC at E; join AE, DE, and through A draw $AF \parallel DE$, and join DF.

Then $\triangle ADE = \triangle DEF$; *I.* 37

$\therefore \triangle ADE + \triangle DBE = \triangle DEF + \triangle DBE$;

$\therefore \triangle ABE = \triangle DBF$.

But $\triangle ABE$ is half of $\triangle ABC$; *I.* 38, *Cor.*

$\therefore \triangle DBF$ is half of $\triangle ABC$.

PROPOSITION 39.

1. Let ABC be a triangle (fig. to App. I. 1), L and K the middle points of AB and AC.

Join BK, CL.

Then $\triangle BLK = \triangle ALK$, and $\triangle CLK = \triangle ALK$; *I.* 38

$\therefore \triangle BLK = \triangle CLK$; $\therefore LK$ is $\parallel BC$. *I.* 39

The rest of the proof is the same as in App. I. 1.

2. Let ABC be a triangle, right-angled at C; and let L, the middle point of the hypotenuse AB, be joined to C.

Bisect AC at K, and join KL.

Then LK is $\parallel BC$, by the preceding deduction;

$\therefore \angle AKL = \angle ACB$; *I.* 29

$\therefore \angle AKL$ and $\angle CKL$ are right. *I.* 13

In $\triangle$s AKL, CKL, $\left\{\begin{array}{c} AK = CK \\ KL = KL \\ \angle AKL = \angle CKL; \end{array}\right.$

$\therefore AL = CL$. *I.* 4

3. Let $ABCD$ be a quadrilateral, E, F, G, H the middle points of AB, BC, CD, DA.

Join AC, BD.

Then EF and GH are each $\parallel AC$, and = half of AC, by the first deduction from I. 39;

$\therefore EF$ is $\parallel GH$ and $= GH$.

Hence also FG is $\parallel$ HE and $= HE$;

$\therefore$ $EFGH$ is a $\parallel^m$.

Again $EF + GH = AC$, by the first deduction from I. 39;

and $\qquad FG + HE = BD$, for the same reason;

$\therefore$ $EF + FG + GH + HE = AC + BD$.

If AC, BD intersect each other at O perpendicularly, then the angles E, F, G, H of the $\parallel^m$ may be proved to be respectively = the angles at O; *I.* 34

$\therefore$ the $\parallel^m$ $EFGH$ is a rectangle.

If AC be $= BD$, then EF and GH, being each half of AC, will be $= FG$ and HE, which are each half of BD;

$\therefore$ the $\parallel^m$ $EFGH$ is a rhombus.

If AC be $= BD$, and if AC, BD intersect each other perpendicularly, then the $\parallel^m$ $EFGH$ is both a rhombus and a rectangle; that is, it is a square.

4. Let ABC, DBC be two equal triangles on opposite sides of the same base BC, and let AD, BC intersect at E.

   If AE be greater than DE, then $\triangle\ AEB$ is greater than $\triangle\ DEB$, and $\triangle\ AEC$ is greater than $\triangle\ DEC$, by the first deduction from I. 38;

   $\therefore$ $\triangle\ AEB + \triangle\ AEC$ is greater than $\triangle\ DEB + \triangle\ DEC$;

   $\therefore$ $\triangle\ ABC$ is greater than $\triangle\ DBC$, which is impossible.

   Hence also DE is not greater than AE;

   $\therefore$ $AE = DE$.

5. Let A be the given point, BC the given straight line.

   Join BA, and produce it its own length to D;

   join CD, and bisect it at E.

   Then AE is $\parallel$ BC.

6. It is proved in I. 16 that $\triangle\ AEB = \triangle\ CEF$;

   $\therefore$ $\triangle\ AEB + \triangle\ CEB = \triangle\ CEF + \triangle\ CEB$;

   $\therefore$ $\triangle\ ABC = \triangle\ FBC$;

   $\therefore$ AF is $\parallel$ BC. *I.* 39

7. Let $ABCD$ be a quadrilateral, AC, BD its diagonals.

   Because $\triangle\ ABC = \triangle\ DCB$, since each is half of the quadrilateral $ABCD$,

   $\therefore$ AD is $\parallel$ BC. *I.* 39

   Hence also AB is $\parallel$ CD, and $ABCD$ is a $\parallel^m$.

8. Let ABC be the given triangle.

   Bisect BC, CA, AB at H, K, L; join HK, KL, LH.

   See the fifteenth deduction from I. 34.

PROPOSITION 40.

1. For $\triangle ABC = \triangle AEF$; *I.* 38
   $\therefore$ $\triangle AEF = \triangle DEF$;
   $\therefore$ AD is $\parallel$ BF. *I.* 39
2. For $\triangle DEF = \triangle DBC$; *I.* 38
   $\therefore$ $\triangle ABC = \triangle DBC$;
   $\therefore$ AD is $\parallel$ BF. *I.* 39
3. Let A, B, C be the vertices of any three of the triangles.
   Then if A, B, C be not in the same straight line, through B there will be drawn two straight lines AB, BC, both of which are parallel to the straight line in which the bases lie, which is impossible. *I. Ax.* 11
4. Let ABC, DEF be equal triangles between the same parallels AD, BF.
   The base BC cannot be greater than the base EF,
   for then $\triangle ABC$ would be greater than $\triangle DEF$, by the first deduction from I. 38.
   For a similar reason, EF cannot be greater than BC;
   $\therefore$ $BC = EF$.
5. Let $ABCD$, $EBCF$ be two trapeziums on the same base BC, and between the same parallels AF, BC; and let AD be $= EF$.
   Join BD, CE.
   Then $\triangle ABD = \triangle ECF$, and $\triangle DBC = \triangle EBC$; *I.* 38
   $\therefore$ $\triangle ABD + \triangle DBC = \triangle ECF + \triangle EBC$;
   $\therefore$ trapezium $ABCD$ = trapezium $EBCF$.
6. Let ABC be a triangle, AH the median from the vertex to the base BC; let DE be $\parallel$ BC, meeting AB, AC at D, E, and the median AH at F.
   Then $\triangle ABH = \triangle ACH$, and $\triangle DBH = \triangle ECH$; *I.* 38
   $\therefore$ $\triangle ABH - \triangle DBH = \triangle ACH - \triangle ECH$;
   $\therefore$ $\triangle ADH = \triangle AEH$;
   $\therefore$ DE is bisected by AH, by the fourth deduction from I. 39.
7. Let DE be the given straight line.
   Draw any straight line $BC \parallel DE$; take any point H in BC, and make $HC = BH$. Produce BD, CE to meet at A, and join AH, cutting DE in F.
   DE is bisected at F.

PROPOSITION 41.

1. Through C draw $CF \parallel BE$, meeting AE produced at F.
   Then $\|^m ABCD = \|^m EBCF$. *I.* 35
   But $\|^m EBCF$ is twice $\triangle EBC$; *I.* 34
   $\therefore\ \|^m ABCD$ is twice $\triangle EBC$.
2. Let $\|^m ABCD$ and $\triangle EFG$ be on equal bases BC, FG, and between the same parallels AE, BG.
   Join AC.
   Then $\triangle ABC = \triangle EFG$. *I.* 38
   But $\|^m ABCD$ is twice $\triangle ABC$; *I.* 34
   $\therefore\ \|^m ABCD$ is twice $\triangle EFG$.
3. Let $ABCD$ be a $\|^m$, and EFG a triangle between the same parallels AE, BG; and let the base FG of the triangle be double the base BC of the $\|^m$.
   Bisect FG in H, and join EH.
   Then $\|^m ABCD$ is twice $\triangle EFH$, by the preceding deduction;
   and $\triangle EFG$ is twice $\triangle EFH$; *I.* 38, *Cor.*
   $\therefore\ \|^m ABCD = \triangle EFG$.
4. If a $\|^m$ and a triangle which are between the same parallels be equal, the base of the triangle is double that of the $\|^m$.
   Let $ABCD$ a $\|^m$, and EFG a triangle, which are between the same parallels AE, BG, be equal;
   the base FG of the triangle shall be double of BC, the base of the $\|^m$.
   If not, make FH = twice BC, and join EH.
   Then $\|^m ABCD = \triangle EFH$, by the preceding deduction.
   But $\|^m ABCD = \triangle EFG$;
   $\therefore\ \triangle EFH = \triangle EFG$, which is impossible;
   $\therefore\ FG$ = twice BC.
5. Let $ABCD$ be a $\|^m$, and let O, a point within it, be joined to A, D, B, C.
   Through O draw $EF \parallel AD$ or BC, meeting AB, CD in E, F.
   Then $\triangle OAD = \frac{1}{2} \|^m AEFD$, and $\triangle OBC = \frac{1}{2} \|^m EBCF$; *I.* 41
   $$\therefore\ \triangle OAD + \triangle OBC = \tfrac{1}{2} \|^m AEFD + \tfrac{1}{2} \|^m EBCF,$$
   $$= \tfrac{1}{2} \|^m ABCD.$$

   The theorem is true when O is taken outside the $\|^m$, provided O lie anywhere between AD and BC produced. If O lie anywhere else, it is the difference of the $\triangle$s OAD, OBC instead of the sum, which is equal to half the $\|^m$.

6. Let EF and GH drawn through A and C be $\parallel$ BD, FG and HE drawn through B and D be $\parallel$ AC.
   Then $\triangle\ ABD = \frac{1}{2}\parallel^{m}EFBD$, and $\triangle\ CBD = \frac{1}{2}\parallel^{m}BGHD$; *I.* 41
   $\therefore\quad \triangle\ ABD + \triangle\ CBD = \frac{1}{2}\parallel^{m}EFBD + \frac{1}{2}\parallel^{m}BGHD$;
   $\therefore$ quadrilateral $ABCD = \frac{1}{2}\parallel^{m}EFGH$.
7. Let the diagonals of the quadrilateral $ABCD$ (fig. to preceding deduction) intersect at O.
   Join EG.
   Then $\triangle\ EFG = \frac{1}{2}\parallel^{m}EFGH$. *I.* 34
   But $ABCD = \frac{1}{2}\parallel^{m}EFGH$, by the preceding deduction;
   $\therefore\quad ABCD = \triangle\ EFG$.
   Now $\triangle\ EFG$ has its sides EF, FG respectively equal to BD, AC, and $\angle\ EFG = \angle\ AOB$. *I.* 34
   The second part of the deduction follows from the fact that each of the quadrilaterals = a triangle which has two sides respectively equal to the diagonals, and the contained angle equal to any of the angles at the intersection of the diagonals.

PROPOSITION 42.

1. Make $\angle\ D$ right, and repeat the construction and proof of the proposition.
2. Let $FECG$ be the given $\parallel^{m}$ (fig. to I. 42), D the given angle.
   Produce CE its own length to B; at B make $\angle\ CBA = \angle\ D$, and let BA meet GF or GF produced at A. Join AC.
   ABC is the required triangle.
3. Make $\angle\ D$ right, and repeat the construction of the preceding deduction.
4. Make a $\parallel^{m}$ = the given triangle, and then, by the fourth deduction from I. 35, construct a rhombus = the $\parallel^{m}$.

PROPOSITION 43.

1. EF divides $ABCD$ into the $\parallel^{ms}$ $AEFD$, $EBCF$.
   GH divides $ABCD$ into the $\parallel^{ms}$ $ABGH$, $HGCD$.
   EF and GH divide $ABCD$ into the $\parallel^{ms}$ $AEKH$, $KGCF$, $EBGK$, $HKFD$.

The arms of the angles of every one of the eight $\|^{ms}$ are either parallel to the arms of the angles of $ABCD$, or coincident with them; $\therefore$ the angles are equal. *I.* 34, *Cor.*

2. Complement BK = complement KD; *I.* 43
   $\therefore BK + EH = KD + EH$;
   $\therefore \quad \|^m AG = \|^m ED.$
   Again $BK + GF = KD + GF$;
   $\therefore \quad \|^m BF = \|^m DG.$

3. If the diagonal AC do not pass through K, let it cut KG at L; through L draw $MN \parallel AD$, cutting AB and CD at M and N.
   Then complement BL = complement LD; *I.* 43
   $\therefore BL$ is greater than KD,
   much more then is BK greater than KD,
   which is impossible;
   $\therefore AC$ must pass through K.

4. By the eighth deduction from I. 27, a rhombus is a $\|^m$.
   Let $ABCD$ be a rhombus (fig. to I. 43).
   Because HK is $\parallel DC$, $\therefore \angle HKA = \angle DCA$, *I.* 29
   $= \angle DAC$; *I.* 5
   $\therefore HK = HA.$ *I.* 6
   Hence by the first deduction from I. 34, $AEKH$ is a rhombus.

5. The proof of this deduction is the first part of the proof of II. 4.

6. The proof of this deduction is the first part of the proof of II. 7.

7. If EK and KG (fig. to prop.) be respectively $= HK$ and KF, the complements BK and KD will be equal in all respects, for then all their sides and angles will be equal, each to each. Now when $ABCD$ is a rhombus, $EK = HK$ and $KG = KF$, by the fourth deduction from I. 43.

PROPOSITION 44.

1. Make $\angle D$ right, and repeat the construction and proof of the proposition.

2. Let AB be the given straight line, C the given angle, $DEFG$ the given $\|^m$.

Produce EF its own length to H, and join DH.

Then $\triangle\ DEH = \|^m\ DEFG$, by the third deduction from I. 41.

Bisect AB at K, and on AK describe a $\|^m\ LAKM = \triangle\ DEH$, and having $\angle\ LAK = \angle\ C$. *I.* 44

Join LB.

Then $\triangle\ LAB = \|^m\ LAKM$, by the third deduction from I. 41,

$$= \triangle\ DEH,$$
$$= \|^m\ DEFG;$$

and $\angle\ LAB = \angle\ C$.

3. Let AB be the given straight line.

On AB construct, by the preceding deduction, a $\triangle\ LAB =$ the given $\|^m$. If K be the middle point of AB, from K draw $KN \perp AB$, and meeting a parallel to AB through L at the point N. Join NA, NB.

Then $\triangle\ NAB$ is isosceles, by the sixth deduction from I. 4;

and $\triangle\ NAB = \triangle\ LAB$, *I.* 37

$= $ the given $\|^m$.

4. Let ABC be the given triangle, D the given area.

Convert D into an equivalent triangle H; *I.* 37, *Cor.*

bisect AB at E, and on EB describe a $\|^m\ EBFG = H$,

and having $\angle\ EBF = \angle\ ABC$; *I.* 44

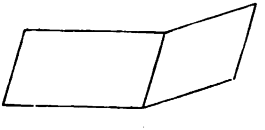
Join AF.

Then $\triangle\ ABF = \|^m\ EBFG$, by the third deduction from I. 41

$= H = D$.

PROPOSITION 45.

1. Yes.



2. Make $\angle\ E$ right, and repeat the construction and proof of the proposition.

3. Let AB be the given straight line.

Describe a rectangle = the given rectilineal figure, by the preceding deduction; convert the rectangle into an equivalent triangle; and on AB describe a rectangle = the triangle, by the first deduction from I. 44.

4. The given area of the rectangle is assumed to be some rectilineal figure; this deduction, therefore, reduces to the preceding.
5. By I. 37, Cor. convert the given rectilineal figure into an equivalent triangle; then apply I. 42.
6. Let A and B be the two given rectilineal figures.
   Make a $\|^m$ $CDEF = A$, and on FE, on the opposite side from CD, construct a $\|^m$ $FEGH = B$, and having $\angle FEG = \angle D$.
   Then as in I. 45, $CDGH$ is a $\|^m$, and it is $= A + B$.
7. Let A and B be the two given rectilineal figures, A being the greater.
   Make a $\|^m$ $CDEF = A$, and on FE, on the same side as CD, construct a $\|^m$ $FEGH = B$, and having $\angle FEG =$ the supplement of $\angle D$.
   Then EG and FH will fall on ED and FC; and $CDGH$ will be a $\|^m = A - B$.

PROPOSITION 46.

1. It would have been sufficient to say that a square is a quadrilateral which has all its sides equal and one of its angles a right angle; the other angles could then have been proved to be right angles.
2. Let $ABCD$, $EFGH$ be two equal squares.
   If AB be not $= EF$, let AB be the greater.
   From BA, BC cut off BK, BL each $= EF$; join KL, AC, EG.

   In $\triangle$s KBL, EFG, $\left\{\begin{array}{l} KB = EF \\ BL = FG \\ \angle B = \angle F; \end{array}\right.$

   $\therefore \triangle KBL = \triangle EFG$, *I. 4*
   $= \frac{1}{2} EFGH$,
   $= \frac{1}{2} ABCD$,
   $= \triangle ABC$, which is impossible;
   $\therefore AB = EF$.
3. AC is $\|$ BD; *I. 28, Cor.*
   $\therefore ABDC$ is a $\|^m$. *I. 33*
   Because $AB = AC$, $\therefore$ all the sides of $\|^m$ $ABDC$ are equal, by the first deduction from I. 34;
   and because $\angle A$ is right, $\therefore$ all the angles of $\|^m$ $ABDC$ are right, by the third deduction from I. 34.

4. By construction $ABDC$ is a rhombus;
   $\therefore$ it is a $\|^{m}$, by the eighth deduction from I. 27.
   But since $\angle A$ is right,
   $\therefore$ all the angles are right, by the third deduction from I. 34.

5. Let AC be the given diagonal.
   Bisect AC at E; from E draw $BED \perp AC$, and make EB, ED each $= AE$ or EC; join AB, BC, CD, DA.
   Then $AB = BC$, by the sixth deduction from I. 4;
   hence also $BC = CD$, $CD = DA$;
   $\therefore ABCD$ is equilateral.
   By the first deduction from I. 32, $\angle$s ABE, EBC are each half a right angle; $\therefore \angle ABC$ is right.
   Hence also, $\angle$s BCD, CDA, DAB are right.

PROPOSITION 47.

1. Let A and B be sides of the two given squares.
   Construct a right-angled triangle with A as base and B as perpendicular, by the eighth deduction from I. 23. The hypotenuse of this right-angled triangle will be a side of the square required.

2. Let A, B, and C be sides of the three given squares.
   Find, by the preceding deduction, D a side of the square equal to the sum of the squares on A and B;
   then find E a side of the square equal to the sum of the squares on D and C.

3. Let A and B be sides of the two given squares.
   Construct a right-angled triangle with the less of the two as base and the greater as hypotenuse. The other side of this right-angled triangle will be a side of the square required.

4. Let A be a side of the given square.
   Take a straight line $B = A$, and apply the first deduction from I. 47.

5. Let $ABDC$ (fig. to I. 46) be the given square.
   Join AD, BC intersecting in E.
   Then, by the tenth deduction from I. 4, and the sixth deduction from I. 10, AE and EB are equal, and $\angle AEB$ is right;
   $\therefore AB^2 = AE^2 + EB^2 = 2\,AE^2$;
   $\therefore AE$ is a side of the square required.

6. Let A be a side of the given square.
Take two straight lines B and C each $= A$, and apply the second deduction from I. 47.

7. For AD^2 (see fig. to the fifth deduction from I. 47) $= AB^2 + BD^2 = 2\,AB^2$.

8. By the fifth deduction from I. 47 (see fig. thereto) $AB^2 = 2\,AE^2$;
and by the preceding deduction $AD^2 = 2\,AB^2$;
$\therefore\ AD^2 = 4\,AE^2$.
Now $AD = 2\,AE$, by the tenth deduction from I. 29.

9. Let $ABCD$ be the given rectangle, AC and BD its diagonals.
Then $AC^2 = AB^2 + BC^2$, and $BD^2 = BC^2 + CD^2$;
$$\therefore\ AC^2 + BD^2 = AB^2 + 2\,BC^2 + CD^2,$$
$$= AB^2 + BC^2 + CD^2 + DA^2.$$

10. Let $ABCD$ be a rhombus whose diagonals AC, BD intersect at E.
Then by the sixth deduction from I. 10, $AE = EC$, $BE = ED$, and the angles at E are right;
$\therefore\ AB^2 + BC^2 + CD^2 + DA^2$
$= AE^2 + EB^2 + BE^2 + EC^2 + CE^2 + ED^2 + DE^2 + EA^2$,
$= 4\,AE^2 + 4\,BE^2$,
$= AC^2 + BD^2$, by the eighth deduction from I. 47.

11. Let ABC and DEF be two triangles having $AB = DE$, $BC = EF$, and $\angle$s C and F right.
Then $AB^2 = AC^2 + CB^2$, and $DE^2 = DF^2 + FE^2$.
But $AB^2 = DE^2$; $\therefore\ AC^2 + CB^2 = DF^2 + FE^2$.
Now $CB^2 = FE^2$; $\therefore\ AC^2 = DF^2$, and $AC = DF$.
Hence $\triangle$s ABC, DEF are congruent. *I.* 8

12. Let ABC be a triangle, and let AD be drawn $\perp$ BC.
Then $AB^2 - AC^2 = (BD^2 + AD^2) - (CD^2 + AD^2)$,
$$= BD^2 - CD^2.$$

13. Let ABC be a triangle having $\angle\ C$ acute.
From C draw $CD \perp CB$ and $= CA$, and join BD.
$$\text{In } \triangle\text{s } BCD,\ BCA,\ \begin{cases} BC = BC \\ CD = CA \\ \angle\ BCD \text{ is greater than } \angle\ BCA; \end{cases}$$
$\therefore\ BD$ is greater than BA; *I.* 24
$\therefore\ BD^2$ is greater than BA^2.
But $BD^2 = BC^2 + CD^2 = BC^2 + CA^2$;
$\therefore\ BA^2$ is less than $BC^2 + CA^2$.

14. Repeat the previous construction and proof, substituting obtuse for acute, less for greater, and greater for less.

15. Let ABC be a triangle, right-angled at C, and let AH, BK be the medians from A and B.

Then $4\,AH^2 + 4\,BK^2$,
$= 4\,AC^2 + 4\,CH^2 + 4\,BC^2 + 4\,CK^2$,
$= 4\,AC^2 + BC^2 + 4\,BC^2 + AC^2$, by eighth deduction from I. 47,
$= 5\,AC^2 + 5\,BC^2 = 5\,AB^2$.

16. Let ABC be an equilateral triangle with $AD \perp BC$, and therefore bisecting BC, by the third deduction from I. 26.

Then $AB^2 = AD^2 + BD^2 = AD^2 + \frac{1}{4}\,BC^2$, by the eighth deduction from I. 47;
$\therefore\ AB^2 = AD^2 + \frac{1}{4}\,AB^2$;
$\therefore \frac{3}{4}\,AB^2 = AD^2$, or $3\,AB^2 = 4\,AD^2$.

17. Let AB be the given straight line, M a side of the given square.

At B make $\angle\ ABD$ = half a right angle; with A as centre and M as radius describe a circle cutting BD at D; draw $DC \perp AB$. AC and CB are the required parts.
For $\angle\ CBD$ = half a right angle, $\angle\ DCB$ is right;
$\therefore\ \angle\ BDC$ = half a right angle; $\therefore CD = CB$.
Hence $AC^2 + CB^2 = AC^2 + CD^2 = AD^2 = M^2$.
If the circle described with A as centre and M as radius does not meet BD, then the problem is impossible, for the square on M is too small. If the circle meet BD at one or more points such that the perpendiculars from these points to AB do not meet AB, the problem is impossible, for the square on M is too large.

18. Let AB be the given straight line.

At A make $\angle\ BAD$ = half a right angle, and at B make $\angle\ ABD$ = one-fourth of a right angle, and let AD, BD meet at D. From D draw $DC \perp AD$ and meeting AB at C.
AC and CB are the required parts.
For $\angle\ ADB = \frac{5}{4}$ of a right angle; *I.* 32
$\therefore\ \angle\ CDB = \frac{1}{4}$ of a right angle;
$\therefore CD = CB$ and $\angle\ ACD$ = half a right angle; *I.* 6, 32
$\therefore AD = CD$. *I.* 6
Now $AC^2 = AD^2 + CD^2 = 2\,CD^2 = 2\,CB^2$.

19. Let ABC be a triangle having $\angle\ C$ obtuse.

Then, by the fourteenth deduction, AB^2 is greater than $AC^2 + BC^2$. Suppose the lengths of AC and BC to remain constant, but $\angle\ C$ to become more obtuse;
then $AC^2 + BC^2$ remains constant, but AB^2 increases.

Hence when $\angle C$ is as obtuse as possible, that is, when AC and BC are in the same straight line, AB^2 is still greater than $AC^2 + BC^2$,
and AB is a straight line divided into the two parts AC, BC.

20. Let $ABCD$ be the rectangle, P any point.
Through P draw $EPF \parallel AB$ or CD, and consequently $\perp AD$ or BC, and let it meet AD, BC, or these lines produced, at E and F.
Then $PA^2 + PC^2 = PE^2 + AE^2 + PF^2 + CF^2$,
and $PB^2 + PD^2 = PF^2 + BF^2 + PE^2 + DE^2$.
But $AE = BF$, and $CF = DE$; *I.* 34
$\therefore PE^2 + AE^2 + PF^2 + CF^2 = PF^2 + BF^2 + PE^2 + DE^2$;
$\therefore PA^2 + PC^2 = PB^2 + PD^2$.

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21. The square BG and $\triangle FBC$ could not be between the same parallels unless AG and AC were in the same straight line; and the square CH and $\triangle BCK$ could not be between the same parallels unless AB and AH were in the same straight line.

22. For $\angle FAB$ and $\angle KAC$ are each half a right angle by the third deduction from I. 8, and $\angle BAC$ is right;
$\therefore AF$ and AK are in the same straight line. *I.* 14

23. For $\angle BGA$ and $\angle ACH$ are each half a right angle by the third deduction from I. 8, and they are alternate;
$\therefore BG$ is $\parallel CH$. *I.* 27

24. If $\triangle ABD$ be rotated anti-clockwise round B through a right angle, BA will fall on BF, because $\angle ABF$ is right,
and A will fall on F, because $BA = BF$.
And BD will fall on BC, because $\angle DBC$ is right;
and D will fall on C, because $BD = BC$.
Hence AD will coincide with FC, and $\triangle ABD$ will coincide with $\triangle FBC$.
Similarly for $\triangle$s ACE, KCB.

25. Since AD, after being rotated through a right angle, coincides with FC, before rotation AD must have been $\perp FC$.
Similarly for AE and KB.

26. Because the four angles ABC, FBD, ABF, CBD = 4 rt. $\angle$s, *I.* 13, *Cor.* 2
and $\angle$s ABF, CBD are each right;
$\therefore$ $\angle$s ABC, FBD are supplementary.

27. In $\triangle$s FBD, ABC, $\begin{cases} FB = AB \\ BD = BC \\ \angle FBD \text{ is supplementary to } \angle ABC; \end{cases}$

$\therefore \triangle FBD = \triangle ABC$. *I.* 38, *Cor.*

Similarly $\triangle KCE = \triangle ABC$.

28. Let T be the point where FG meets LA.

Then since GA is $\perp AB$, and AT is $\perp BC$,

$\therefore \angle GAT = \angle ABC$, by the nineteenth deduction from I. 32.

Hence $\triangle$s GAT, ABC are congruent, and $AT = BC$. *I.* 26

If T' be the point where KH meets LA, it may similarly be proved that $AT' = BC$;

$\therefore T$ and T' are the same point.

29. It has been proved in the preceding deduction that $\triangle$s GAT, HTA are $= \triangle ABC$.

In $\triangle$s GAH, BAC, $\begin{cases} GA = BA \\ AH = AC \\ \angle GAH = \angle BAC; \end{cases}$ *I.* 15

$\therefore \triangle GAH = \triangle BAC$. *I.* 4

Since $GAHT$ is a $\|^{m}$, $\therefore \triangle GTH = \triangle GAH$. *I.* 34

30. If $\triangle UBD$ be rotated anti-clockwise through a right angle, BD will fall on BC, because $\angle DBC$ is right,

and D will fall on C, because $BD = BC$.

Also since $\angle UBA$ is right, being supplementary to $\angle ABF$;

$\therefore BU$ will fall along BA.

Since D coincides with C, and DU is $\perp BU$, and CA is $\perp BA$;

therefore DU coincides with CA, and $\triangle UBD$ with $\triangle ABC$.

Similarly $\triangle VEC$ may be made to coincide with $\triangle ABC$.

31. Since $DF^2 = DU^2 + UF^2 = AC^2 + (2\,AB)^2$,

$EK^2 = EV^2 + VK^2 = AB^2 + (2\,AC)^2$;

$\therefore DF^2 + EK^2 = 5\,AB^2 + 5\,AC^2 = 5\,BC^2$. *I.* 47

32. $DF^2 + FG^2 + GH^2 + HK^2 + KE^2 + ED^2$

$= AC^2 + (2\,AB)^2 + AB^2 + BC^2 + AC^2 + AB^2 + (2\,AC)^2 + BC^2$,

$= 6\,AC^2 + 6\,AB^2 + 2\,BC^2 = 8\,BC^2$.

33. Since $\angle ABF$ is right,

$\therefore \angle ABI$ is complementary to $\angle FBM$; *I.* 13

$\therefore \angle ABI = \angle BFM$. *I.* 32

Now, since the hypotenuses AB, BF of the right-angled $\triangle$s ABI, BFM are equal,

$\therefore$ $\triangle$ ABI is congruent to $\triangle$ BFM.

Similarly $\triangle$ ACI is congruent to $\triangle$ CKN.

34. $FM = BI$, and $KN = CI$; $\therefore$ $FM + KN = BC$.

$BN = BC + CN = BC + AI = IL + AI = AL$.

Similarly $CM = AL$.

35. If $\triangle$ ABC be rotated anti-clockwise through a right angle, BA will fall on BF, because $\angle$ ABF is right,

and A will fall on F, because $BA = BF$.

And BC will fall along BP, because $\angle$ CBP is right.

Since A coincides with F, and AC is $\perp$ AB, and FP is $\perp$ FB;

$\therefore$ AC coincides with FP, and $\triangle$ ABC with $\triangle$ FBP.

Similarly $\triangle$ ACB may be made to coincide with $\triangle$ KCQ.

36. Since BP and CQ are each $= BC$, $\therefore$ $BP = CQ$;

and BP is $\parallel$ CQ; $\therefore$ $BCQP$ is a $\parallel^m$. *I.* 33

Now since $\angle$ PBC and $\angle$ BCQ are right,

$\therefore$ all the angles are right;

and since $PB = BC = CQ$, $\therefore$ all the sides are equal. *I.* 34

37. Since BP and AT are each $= BC$, $\therefore$ $BP = AT$;

and BP is $\parallel$ AT; $\therefore$ $ABPT$ is a $\parallel^m$; *I.* 33

and $\parallel^m$ $ABPT = \parallel^m$ BL. *I.* 36

Similarly $ACQT$ is a $\parallel^m$, and $= \parallel^m$ CL.

38. AT and DB are parallel, and equal, since each $= BC$;

$\therefore$ $ADBT$ is a $\parallel^m$; *I.* 33

and $\parallel^m$ $ADBT = \parallel^m$ BL. *I.* 36

Similarly $AECT$ is a $\parallel^m$, and $= \parallel^m$ CL.

39. Since FP and DU are each $= AC$, $\therefore$ $FP = DU$;

and FP is $\parallel$ DU, since each is $\perp$ FU; $\therefore$ $DFPU$ is a $\parallel^m$.

Now $\parallel^m$ $DFPU = 2$ $\triangle$ $DUF = 4$ $\triangle$ $DUB = 4$ $\triangle$ ACB.

Similarly for $EKQV$.

40. Since AH and DU are each $= AC$, $\therefore$ $AH = DU$;

and AH is $\parallel$ DU, since each is $\perp$ BU; $\therefore$ $ADUH$ is a $\parallel^m$.

Now $\parallel^m$ $ADUH = 2$ $\triangle$ BDU (I. 41) $= 2$ $\triangle$ ABC.

Similarly for $AEVG$.

41. AE was proved $\perp$ BK, by the twenty-fifth deduction from I. 47, and AE was proved $\parallel$ CT, by the thirty-eighth deduction from I. 47;

$\therefore$ BK is $\perp$ CT.

Similarly CF is $\perp$ BT.

42. In $\triangle$ TBC, TL, BK, CF are the perpendiculars from its vertices on the opposite sides;

$\therefore$ they are concurrent. *App. I.* 3

43. For $\triangle\ HAX = \triangle\ KAX$. *I.* 37
Add to each $\triangle\ BAX$; $\therefore\ \triangle\ BHX = \triangle\ BAK$.
But $\triangle\ BAK = \triangle\ BAC$ (*I.* 37); $\therefore\ \triangle\ BHX = \triangle\ BAC$.
Similarly for $\triangle\ CGW$.

44. Since the equal $\triangle$s WCG, XBH stand on equal bases CG, BH,
$\therefore$ their altitudes must be equal (*I.* 39); $\therefore\ AW = AX$.

45. Since $\triangle\ CWG = \triangle\ CAB$; and $\triangle\ WAG = \triangle\ XAB$; *I.* 4
$\therefore\ \triangle\ CWG - \triangle\ WAG\ =\ \triangle\ CAB - \triangle\ XAB$;
$\therefore\ \triangle\ ACW = \triangle\ BCX$.
Similarly $\triangle\ ABX = \triangle\ BCW$.

46. Since $\triangle\ ABX = \triangle\ BCW$,
$\therefore\ \triangle\ ABX - \triangle\ BOW = \triangle\ BCW - \triangle\ BOW$;
$\therefore\ AWOX = \triangle\ BOC$.

47. If $\triangle\ ABC$ be rotated clockwise round A through a right angle,
AB will fall on AG, since $\angle\ BAG$ is right;
and B will fall on G, since $AB = AG$.
And AC will fall along AY, since $\angle\ CAY$ is right;
and BC will fall along GR, since GR was drawn $\perp\ BC$.
Hence $\triangle\ BAC$ will coincide with $\triangle\ GAY$.
Similarly for $\triangle\ ZAH$.

48. Since $AZ = AB = BU$, and AZ is $\parallel\ BU$; $\therefore\ UZ$ is $\parallel\ BA$. *I.* 33
But DU is $\parallel\ BA$, since each is $\perp\ FU$;
$\therefore\ DU$ produced passes through Z.
Similarly EV produced passes through Y.
Join GW, VW.
Since $AY = CV = AC$, and $AW = AX$,
$\therefore\ WY = XC$; and $YV = AC = CK$;
$\therefore\ \triangle$s WYV, XCK are congruent (I. 4), and $\angle\ VWY = \angle\ KXC$.
But $\triangle$s WAG, XAB are congruent; *I.* 4.
$\therefore\ \angle\ GWA = \angle\ BXA$.
Now $\angle\ KXC = \angle\ BXA$ (I. 15); $\therefore\ \angle\ VWY = \angle\ GWA$;
$\therefore$ by the second deduction from I. 14, GW, VW are in one straight line.
Similarly HX, UX are in one straight line.

49. It may be proved, as in some of the preceding deductions, that if $\triangle\ ABI$ be rotated clockwise round A through a right angle, it will coincide with $\triangle\ AGG'$; if it be rotated anti-clockwise round A through a right angle, it will coincide with $\triangle\ AZH'$. Also that if $\triangle\ ACI$ be rotated clockwise round A through a right angle, it will coincide with $\triangle\ AYG'$; if it be rotated

anti-clockwise round A through a right angle, it will coincide with AHH'.

50. Since by preceding deduction $AG' = AI$ and $AH' = AI$,

$\therefore \quad AG' = AH'$;

$\therefore \quad IR = IS.$ *I.* 34

$$GR + HS = GG' + HH' + G'R + H'S,$$
$$= BI + CI + AI + AI,$$
$$= BC + 2AI.$$

But $MN = BC + BM + CN = BC + AI + AI$;

$\therefore GR + HS = MN.$

$FM + KN = BC$, by the thirty-fourth deduction;

and $GR + HS = BC + 2AI$;

$\therefore FM + GR + HS + KN = 2(BC + AI).$

If the quadrilateral $AGRB$ be rotated anti-clockwise round A through a right angle, it will be found to take the position $ABSZ$, since $AG = AZ$; $\therefore GR = BS$.

Since the quadrilateral $AGRB$ is congruent to the quadrilateral $ABSZ$,

$\therefore BR = ZS$; and $CB = HZ$, by the forty-seventh deduction;

$\therefore CR = HS.$

PROPOSITION 48.

1. No. Because in the proof of the proposition it is stated that $AD^2 + AC^2 = CD^2$ (I. 47); and this is so only if $\angle DAC$ be right. Now if BA be produced to D, $\angle DAC$ can be right only if $\angle BAC$ is right, and that is what is to be proved.

2. If $\angle BAC$ be not right, from A draw $AD \perp AC$, on the same side as AB, and $= AB$, and join CD.

   Then, as in the text, CD is proved $= CB$;
   therefore on the same base AC and on the same side of it there are two triangles BAC, DAC having $AB = AD$, and $CB = CD$, which is impossible. *I.* 7
   Hence $\angle BAC$ must be right.

3. Let ABC be a triangle, and let AB^2 be less than $BC^2 + CA^2$; $\angle C$ must be acute.

   For if $\angle C$ were right, AB^2 would be $= BC^2 + CA^2$; *I.* 47
   and if $\angle C$ were obtuse, AB^2 would be greater than

$BC^2 + CA^2$, by the fourteenth deduction from I. 47.
Hence $\angle C$ must be acute.

4. Let ABC be a triangle, and let AB^2 be greater than $BC^2 + CA^2$; $\angle C$ must be obtuse.

For if $\angle C$ were right, AB^2 would be $= BC^2 + CA^2$; *I.* 47
and if $\angle C$ were acute, AB^2 would be less than $BC^2 + CA^2$, by the thirteenth deduction from I. 47.
Hence $\angle C$ must be obtuse.

5. $5^2 = 3^2 + 4^2$, because $25 = 9 + 16$; hence the angle opposite the side 5 must be right.

6. Let AB be the given straight line, B the given point in it.
Draw any other straight line, and measure off on it 5 equal parts. With B as centre and radius equal 3 of these parts cut AB at C; with B as centre and radius equal 4 of these parts describe an arc of a circle; with C as centre and radius equal 5 of these parts, describe an arc cutting the former arc in D. DB is $\perp AB$.

For $CD^2 = BC^2 + BD^2$, since $5^2 = 3^2 + 4^2$;
$\therefore \angle B$ is right. *I.* 48

7. (*a*) Let $2n + 1$ denote the less side about the right angle; then the greater side about the right angle will be $2n^2 + 2n$, and the hypotenuse $2n^2 + 2n + 1$.
Now since $(2n^2 + 2n + 1)^2 = 4n^4 + 8n^3 + 8n^2 + 4n + 1$,
and $(2n^2 + 2n)^2 + (2n + 1)^2 = 4n^4 + 8n^3 + 8n^2 + 4n + 1$;
$\therefore$ if integral values be assigned to n in the expressions $2n + 1$, $2n^2 + 2n$, $2n^2 + 2n + 1$, numbers will be obtained representing the sides of right-angled triangles.

(*b*) Let $2n$ denote one of the sides about the right angle; then the other side about the right angle will be $n^2 - 1$, and the hypotenuse $n^2 + 1$.
Now since $(n^2 + 1)^2 = n^4 + 2n^2 + 1$,
and $(n^2 - 1)^2 + (2n)^2 = n^4 + 2n^2 + 1$;
$\therefore$ if integral values be assigned to n in the expressions $2n$, $n^2 - 1$, $n^2 + 1$, numbers will be obtained representing the sides of right-angled triangles.

To show that the two rules are fundamentally the same, denote the less side about the right angle by n; then according to the first rule the other sides will be $\dfrac{n^2 - 1}{2}$ and $\dfrac{n^2 + 1}{2}$.
These values are the halves of the values $2n$, $n^2 - 1$, and $n^2 + 1$ obtained by the second rule.

DEDUCTIONS.

1. Let $ABCD$ be a trapezium whose parallel sides are AD, BC, AD being the less, and let E be the middle point of AB, and F the middle point of CD.

   Join AC, DB, AF, DE.

   Then $\triangle\ ABD = \triangle\ ADC$. *I.* 37

   Now $\triangle\ AED$ = half of $\triangle\ ABD$, and $\triangle\ DFA$ = half of $\triangle\ ADC$; *I.* 38

   $\therefore\ \triangle\ AED = \triangle\ DFA$, and EF is $\parallel AD$. *I.* 39

   Through F draw $GH \parallel AB$, meeting AD at G and BC at H.

   Then from the congruency of $\triangle$s DFG, CFH we have $DG = CH$.

   Hence $AD + BC = (AG - DG) + (CH + BH)$,
   $= AG + BH = 2\ EF$.

2. Let $ACBD$ be a trapezium whose parallel sides are AD, CB, AD being the greater, and whose diagonals are AB, CD. Let E be the middle point of AB, and F the middle point of CD.

   Join AF, DE.

   Then $\triangle\ ABD = \triangle\ ADC$. *I.* 37

   Now $\triangle\ AED$ = half of $\triangle\ ABD$, and $\triangle\ DFA$ = half of $\triangle\ ADC$; *I.* 38

   $\therefore\ \triangle\ AED = \triangle\ DFA$, and EF is $\parallel AD$. *I.* 39

   Through F draw $GH \parallel AB$, meeting AD at G and BC at H.

   Then from the congruency of $\triangle$s DFG, CFH we have $DG = CH$.

   Hence $AD - BC = (AG + DG) - (CH - BH)$,
   $= AG + BH = 2\ EF$.

3. Let $ABCD$ be a trapezium whose parallel sides are AD, BC, and whose diagonals are AC, BD. Let E be the middle point of AB, F the middle point of CD, and let EF meet AC, BD at G and H.

   Then EF is $\parallel BC$ by the first deduction.

   Now since, in $\triangle\ ABC$, EF is drawn through E, the middle point of AB, parallel to BC,

   $\therefore$ EF bisects AC at G. *App. I.* 1, *Cor.* 1

   Similarly EF bisects BD at H.

4. Let $ABCD$ be a quadrilateral, AC, BD its diagonals; and let E, F, G, H, K, L be the middle points of AB, BC, CD, DA, AC, BD respectively.

Then in $\triangle\ ABC$, EK is $\parallel BC$ and $=$ half of BC,
and in $\triangle\ DBC$, LG is $\parallel BC$ and $=$ half of BC; *App. I.* 1
$\therefore$ $EKGL$ is a $\parallel^m$. *I.* 33

5. In the figure to the previous deduction, $EKGL$ is a $\parallel^m$;
$\therefore$ EG bisects KL, by the tenth deduction from I. 29.
Similarly $HKFL$ is a $\parallel^m$ and HF bisects KL;
$\therefore$ EG, HF, KL are concurrent.

6. Let ABC be a triangle, AH the median from A, and G the centroid. From A, B, C, G, H let there be drawn AA', BB', CC', GG', $HH' \perp XY$, a straight line outside the triangle.

Then $BB' + CC' = 2\,HH'$. *App. I.* 1, *Cor.* 2

Now since $AG = 2\,GH$ (App. I. 4, note), if AG be bisected at P, and PP' be drawn $\perp XY$, and through H there be drawn $HQ \parallel XY$ and meeting AA' in Q, it may be proved that $AA' + 2\,HH' = 3\,GG'$;
$\therefore$ $AA' + BB' + CC' = 3\,GG'$.

If XY cut the triangle so that A, B, G lie on one side of it and C on the other, it will be found that $AA' + BB' - CC' = 3\,GG'$; if XY cut the triangle so that A, B lie on one side of it and C, G on the other, $CC' - AA' - BB' = 3\,GG'$; if XY pass through the centroid G, so that A, B lie on one side of it and C on the other, $AA' + BB' = CC'$.

7. Let XY be the given straight line, A and B the given points situated on the same side of XY.

From A draw $AC \perp XY$, and produce it to A' so that $A'C = AC$; join $A'B$ cutting XY at D.

D is the required point.

Join AD, take any other point E in XY, and join AE, BE, $A'E$.

Then $AD = A'D$, and $AE = A'E$; *I.* 4
$\therefore$ $AD + DB = A'D + DB$, and $AE + EB = A'E + EB$.
Now $A'D + DB$, being $= A'B$, is less than $A'E + EB$; *I.* 20
$\therefore$ $AD + DB$ is less than $AE + EB$.

When A and B are on opposite sides of XY, join AB cutting XY at D. D is the required point.

Take E any other point in XY, and join AE, BE.

Then $AE + EB$ is greater than AB; *I.* 20

and $AD + DB = AB$;

$\therefore$ $AD + DB$ is less than $AE + EB$.

8. Let XY be the given straight line, A and B the given points situated either on the same side or on opposite sides of XY. Join AB, bisect it at C, and draw $CD \perp AB$ and meeting XY at D.

Then $AD = BD$ (I. 4), and the difference between AD and BD is the least possible.

It was intended that this deduction should be: Find a point in a given straight line such that the difference of its distances from two given points may be the greatest possible.

Let XY be the given straight line, A and B the given points situated on the same side of XY.

Join AB, and produce it to cut XY at D.

D is the required point.

Take E any other point in XY, and join AE, BE.

Then $AE \sim EB$ is less than AB; *I. 20, Cor.*

and $AD \sim DB = AB$;

$\therefore$ $AD \sim DB$ is greater than $AE \sim EB$.

When A and B are on opposite sides of XY.

From A draw $AC \perp XY$, and produce it to A' so that $A'C = AC$; join $A'B$ cutting XY at D.

D is the required point.

Join AD, take any other point E in XY, and join AE, BE, $A'E$.

Then $AD = A'D$, and $AE = A'E$; *I. 4*

$\therefore AD \sim DB = A'D \sim DB$, and $AE \sim EB = A'E \sim EB$.

Now $AD' \sim DB$, being equal to $A'B$, is greater than $A'E \sim EB$; *I. 20, Cor.*

$\therefore AD \sim DB$ is greater than $AE \sim EB$.

9. Let ABC, DBC be two triangles on the same base BC and having $CA = CD$, and let $\angle ACB$ be right.

From D draw $DE \perp BC$ or BC produced.

Then DE is less than DC (I. 19, Cor.), and consequently less than AC. Hence the altitude of $\triangle ABC$ is greater than that of $\triangle DBC$;

$\therefore$ $\triangle ABC$ is greater than $\triangle DBC$.

10. Let DAB, EAB be two triangles of equal area standing on the same side of the same base AB, and let DAB have $DA = DB$.

Join DE, which is $\parallel AB$ (I. 39), and produce it to X and Y. From A draw $AC \perp XY$, let AC, BD be produced to meet at A', and join $A'E$.

Then $\angle ADC = \angle DAB$, *I.* 29

$= \angle DBA$, *I.* 5

$= \angle BDE$, *I.* 29

$= \angle A'DC$, *I.* 15

$\therefore$ $A'C = AC$. *I.* 26

Hence from the seventh deduction $AD + DB$ is less than $AE + EB$.

11. Let ABC be a triangle whose base BC passes through a given point P and is there bisected, and let ADE be another triangle having the same vertical angle, and whose base DE also passes through P. Suppose D to fall between A and B, and E on AC produced.

Through C draw $CF \parallel AB$ and meeting DE at F.

Then $\triangle BPD = \triangle CPF$. *I.* 26

Add to each of these equals the figure $ADPC$;

$\therefore$ $\triangle ABC$ = quadrilateral $ADFC$;

$\therefore$ $\triangle ABC$ is less than $\triangle ADE$.

12. Let ABC be an isosceles triangle having $AB = AC$; in AC take any point E, and produce AB to D so that $BD = CE$, and join DE: it is required to prove DE greater than BC.

Through D draw $DF \parallel AC$, and through C draw $CF \parallel ED$, and join BF.

Then $DFEC$ is a $\parallel^{m}$;

$\therefore$ $DF = CE = DB$, and $\angle DBF = \angle DFB$.

Because DF is $\parallel AC$, $\therefore \angle BDF$ is supplementary to $\angle A$. *I.* 29

But $\angle BDF$ is supplementary to $\angle DBF + \angle DFB$; *I.* 32

$\therefore$ $\angle DBF + \angle DFB = \angle A$;

$\therefore$ twice $\angle DBF = \angle A$, or $\angle DBF$ = half of $\angle A$;

$\therefore$ BF is $\parallel$ the bisector of $\angle A$.

Now the bisector of $\angle A$ is $\perp BC$;

$\therefore$ BF is $\perp BC$.

Hence FC, and consequently DE, is greater than BC.

13. Let ABC be an isosceles triangle having $AB = AC$, and from P, any point in the base BC, let PE, PF be drawn $\perp$ CA, AB: it is required to prove $PE + PF$ constant.

To find the constant value of $PE + PF$, from B draw $BK \perp CA$. Then if the point P, which may be situated anywhere on BC, coincide with B, the perpendicular PF on

AB vanishes, and the perpendicular PE on CA becomes BK, so that BK is the constant value of $PE + PF$.

To prove $PE + PF = BK$. Produce EP, and from B draw $BM \parallel AC$, and consequently $\perp EM$.

Then $\angle MBP = \angle C$, *I.* 29

$= \angle FBP$; *I.* 5

$\therefore$ $\triangle$s NBP, FBP are congruent, and $PM = PF$. *I.* 26

Hence $PE + PF = PE + PM = BK$, since $BMEK$ is a $\parallel^m$.

If P is situated in CB produced, the same construction and proof will hold, only for $PE + PF$ there must be substituted $PE - PF$.

14. Let ABC be an equilateral triangle, and from P, any point inside the triangle, let PD, PE, PF be drawn $\perp BC, CA, AB$: it is required to prove $PD + PE + PF$ constant.

To find the constant value of $PD + PE + PF$, from A draw $AH \perp BC$. Then if the point P, which may be situated anywhere inside ABC, coincide with A, the perpendiculars PE, PF on CA, AB vanish, and the perpendicular PD on BC becomes AH, so that AH is the constant value of $PD + PE + PF$.

To prove $PD + PE + PF = AH$. Through P draw a parallel to BC meeting AB, AC at B', C' and AH at H'; and from B' draw $B'K' \perp AC'$.

Then $\triangle AB'C'$ is equilateral, $AH' = B'K'$, and $H'H = PD$.

But $PE + PF = B'K'$, by the preceding deduction;

$\therefore$ $PE + PF = AH'$;

$\therefore$ $PD + PE + PF = H'H + AH' = AH$.

If P is situated outside the triangle, but in the region bounded by BA, BC produced, the same construction and proof will hold, only for $PD + PE + PF$ there must be substituted $PD - PE + PF$; and if P is situated outside the triangle, but in the region bounded by BA, CA produced, the same construction and proof will hold, only for $PD + PE + PF$ there must be substituted $PD - PE - PF$.

15. Let ABC be a triangle, AX, BY, CZ the three perpendiculars.

Then $AX + XB$ is greater than AB,

and $AX + XC$ is greater than CA; *I.* 20

$\therefore$ $2\,AX + BC$ is greater than $AB + CA$.

Similarly $2\,BY + CA$ is greater than $AB + BC$.

and $2\,CZ + AB$ is greater than $CA + BC$;

$\therefore$ $2(AX + BY + CZ) + AB + BC + CA$ is greater than $2(AB + BC + CA)$;

$\therefore$ $2(AX + BY + CZ)$ is greater than $AB + BC + CA$;

$\therefore$ $AX + BY + CZ$ is greater than $\frac{1}{2}(AB + BC + CA)$.

Again $2\,AX$ is less than $AB + CA$, *I.* 19, *Cor.*

$2\,BY$ is less than $AB + BC$,

$2\,CZ$ is less than $CA + BC$;

$\therefore$ $2(AX + BY + CZ)$ is less than $2(AB + BC + CA)$;

$\therefore$ $AX + BY + CZ$ is less than $AB + BC + CA$.

16. Let ABC be a triangle, and let AD be $\perp$ BC; if AC is greater than AB, it is required to prove $\angle\, CAD$ greater than $\angle\, BAD$, and CD greater than BD.

Since AC is greater than AB, $\therefore$ $\angle\, B$ is greater than $\angle\, C$;

$\therefore$ complement of $\angle\, B$ is less than complement of $\angle\, C$;

$\therefore$ $\angle\, BAD$ is less than $\angle\, CAD$.

Again since AC is greater than AB,

$\therefore$ AC^2 is greater than AB^2;

$\therefore$ $AD^2 + DC^2$ is greater than $AD^2 + DB^2$; *I.* 47

$\therefore$ DC^2 is greater than DB^2, and DC than DB.

17. Let ABC be a triangle, and let AD bisect $\angle\, BAC$; if AC is greater than AB, it is required to prove CD greater than BD.

From AC cut off $AE = AB$, and join DE.

Then $\triangle$s BAD, EAD are congruent; *I.* 4

$\therefore$ $BD = ED$, and $\angle\, ADB = \angle\, ADE$.

Now $\angle\, DEC$ is greater than $\angle\, ADE$; *I.* 16

$\therefore$ $\angle\, DEC$ is greater than $\angle\, ADB$.

But $\angle\, ADB$ is greater than $\angle\, C$; *I.* 16

$\therefore$ $\angle\, DEC$ is greater than $\angle\, C$;

$\therefore$ DC is greater than DE, that is, than DB.

18. Let ABC be a triangle, and from A let the median AH be drawn; if AC is greater than AB, it is required to prove $\angle\, BAH$ greater than $\angle\, CAH$.

Produce AH to D, making $HD = AH$, and join CD.

Then $\triangle$s ABH, DCH are congruent; *I.* 4

$\therefore$ $AB = DC$, and $\angle\, BAH = \angle\, CDH$.

Now since AC is greater than AB, $\therefore$ AC is greater than DC;

$\therefore$ $\angle\, CDH$ is greater than $\angle\, CAH$;

$\therefore$ $\angle\, BAH$ is greater than $\angle\, CAH$.

19. Let ABC be a triangle, and from A let there be drawn AD $\perp$ BC, AE bisecting $\angle\, BAC$, and AF bisecting BC: it is required to prove that AE lies between AD and AF.

If $AB = AC$, the three straight lines AD, AE, AF coincide. Let therefore AC be greater than AB.

Since AD is $\perp BC$,

$\therefore \angle CAD$ is greater than $\angle BAD$, by the 16th deduction;

$\therefore AE$, the bisector of $\angle BAC$, lies between AD and AC.

Since AF is a median,

$\therefore \angle BAF$ is greater than $\angle CAF$ by the eighteenth deduction;

$\therefore AE$, the bisector of $\angle BAC$, lies between AF and AB.

Hence AE lies in position between AD and AF;

$\therefore$ AE lies in magnitude between AD and AF. *I.* 19, *Cor.*

20. Since the sum of the perpendiculars from the vertices of a triangle on the opposite sides is greater than the semiperimeter, by the fifteenth deduction; and since, by the preceding deduction, the sum of the angular bisectors is never less than the sum of the perpendiculars; the first part of the theorem is proved.

Again since the sum of the three medians is less than the perimeter, by the fourteenth deduction from I. 20; and since, by the preceding deduction, the sum of the angular bisectors is never greater than the sum of the medians; the second part of the theorem is proved.

21. Let ABC be a triangle, having AC greater than AB; from B and C let BD and CE be drawn $\perp AC$ and AB respectively.

From CA cut off $CF = BE$, and join BF.

Because AC is greater than AB,

$\therefore \angle EBC$ is greater than $\angle FCB$.

In $\triangle$s EBC, FCB, $\begin{cases} EB = FC \\ BC = CB \\ \angle EBC \text{ is greater than } \angle FCB; \end{cases}$

$\therefore EC$ is greater than FB. *I.* 24

But FB is greater than DB; *I.* 19, *Cor.*

$\therefore EC$ is greater than DB.

[In this proof it is assumed that F does not coincide with D. If it were possible for F to fall on D, EC would still be greater than BD.]

22. Let ABC be a triangle having AC greater than AB, and let BK, CL be the medians drawn to them, and intersecting at G.

Draw AH the third median, also passing through G. *App. I.* 4

Then BG is less than CG, by the seventh deduction from I. 25.

Now BG is two-thirds of BK, and CG is two-thirds of CL;
$\therefore$ BK is less than CL.

23. Let ABC be a triangle having AC greater than AB, and let BE, CF, the bisectors of $\angle$s ABC, ACB, meet AC, AB in E and F.
Because AC is greater than AB,
$\therefore$ $\angle ABC$ is greater than $\angle ACB$;
$\therefore$ $\angle EBC$ is greater than $\angle FCB$, and $\angle ABE$ greater than $\angle ACF$.
Make $\angle EBH = \angle ACF$, and let BH meet AE at H, and CF at G.
Because $\angle EBC$ is greater than $\angle FCB$,
and $\angle EBH$ is equal to $\angle ACF$;
$\therefore$ $\angle HBC$ is greater than $\angle HCB$;
$\therefore$ HC is greater than HB.
Produce BH, and make $BK = CH$;
through K draw $KL \parallel AC$ and meeting BE produced at L.

In $\triangle$s BKL, CHG, $\left\{\begin{array}{l} \angle KBL = \angle HCG \\ \angle BKL = \angle CHG \\ BK = CH; \end{array}\right.$ *I.* 29

$\therefore$ $BL = CG$. *I.* 26
Now BE is less than BL, and CG less than CF;
$\therefore$ BE is less than CF.
[This solution was given by Jacob Thompson Dunne in the *Lady's, Farmer's, and Mathematical Almanack* for 1862 (Dublin), p. 131.]

24. Let ABC (fig. on p. 302 of *Euclid*) be right-angled at A.
From BA cut off $BE = BD$; through E draw $EF \parallel AC$, meeting BC at F, and through F draw $FG \parallel BA$, meeting AC at G.
Then $\triangle$s BEF, BDA are congruent; *I.* 26
$\therefore$ $BF = BA$, and $AD = EF = AG$; *I.* 34
$\therefore$ $BF + AD = BA + AG$.
Now since $\angle FGC = \angle BAC =$ a right angle;
$\therefore$ FC is greater than GC; *I.* 19, *Cor.*
$\therefore$ $BF + FC + AD$ is greater than $BA + AG + GC$.

25. Let ABC be a triangle (fig. on p. 100 of *Euclid*), AH, BK, CL the three medians intersecting at G.
Then $AG + BG$ is greater than AB;
$\therefore$ $\frac{2}{3}(AH + BK)$ is greater than AB.
Hence also $\frac{2}{3}(BK + CL)$ is greater than BC,

and $\frac{2}{3}(CL + AH)$ is greater than CA;

$\therefore$ $\frac{4}{3}(AH + BK + CL)$ is greater than $AB + BC + CA$;

$\therefore$ $AH + BK + CL)$ is greater than $\frac{3}{4}(AB+BC+CA)$.

26. At either end of the perpendicular and on opposite sides of it draw straight lines making angles equal to half of an angle of an equilateral triangle; at the other end of the perpendicular draw a straight line at right angles to the perpendicular and meeting the two other straight lines.

27. Let BAC be the given vertical angle.
Bisect it by AD, and from AD cut off AE = the given perpendicular; through E draw a straight line at right angles to AE, and meeting AB, AC at B and C.
ABC is the required triangle.

28. Let EF be the given perimeter.
Bisect EF at D, and draw $DA \perp EF$ and = the given perpendicular. Join EA, FA; at A make $\angle EAB = \angle E$, and $\angle FAC = \angle F$, and let AB, AC meet EF at B and C.
ABC is the required triangle.

29. Let AB be the given hypotenuse.
At B make $\angle ABD$ = the given acute angle, and from A draw $AC \perp BD$. ABC is the required triangle.

30. Let BC be the given side.
At C draw $CD \perp BC$; with B as centre and a radius equal to the given hypotenuse cut CD at A.
ABC is the required triangle.

31. Let BD be the given sum of sides.
At D make $\angle BDA$ = half of a right angle, and draw $BE \perp BD$ meeting DA at E. With centre B, and radius = the given hypotenuse, describe a circle, cutting DE at A; from A draw $AC \perp BD$. ABC is the required triangle.
For $\angle CAD$ = half of a right angle; $\therefore CA = CD$, and $CA + CB = BD$.
If the circle cut DE at another point A', between D and E, there will be another solution.

32. Let BD be the given difference of sides.
At D make $\angle BDA$ = half of a right angle, and draw $BE \perp BD$, meeting DA at E. With centre B, and radius = the given hypotenuse, describe a circle cutting DE produced at A; from A draw $AC \perp DB$ produced.
ABC is the required triangle.

For $\angle\ CAD$ = half of a right angle; $\therefore CA = CD$, and $CA - CB = BD$.

If the circle cut ED produced at another point A', there will be another solution.

33. Let CD be the perpendicular from the right angle C on the hypotenuse, and BC the given side.

Construct, by the thirtieth deduction, a right-angled triangle BCD of which BC is the hypotenuse and CD one of the sides. From C draw $CA \perp BC$, and produce BD to meet it at A. ABC is the required triangle.

34. Let CD be the perpendicular from the right angle C on the hypotenuse, and CE the median.

Construct, by the thirtieth deduction, a right-angled triangle CDE of which CE is the hypotenuse and CD one of the sides. Produce DE both ways to A and B, making EA and EB each $= EC$, and join AC, BC.

ABC is the required triangle.

For $\angle\ ECA = \angle\ A$, and $\angle\ ECB = \angle\ B$; *I.* 5

$\therefore \angle\ ACB = \angle\ A + \angle\ B$;

$\therefore \angle\ ACB$ is right.

35. Let BD be the given sum of sides.

At B make $\angle\ DBA$ = the given acute angle; at D make $\angle\ BDA$ = half of a right angle, and let BA, DA meet at A. From A draw $AC \perp BD$. ABC is the required triangle.

For $\angle\ CAD$ = half of a right angle;

$\therefore CA = CD$, and $CB + CA = BD$.

36. Let BD be the given difference of sides.

At B make $\angle\ DBA$ = the given acute angle; at D make $\angle\ BDA$ = the supplement of half of a right angle, and let BA, DA meet at A. From A draw $AC \perp BD$ produced.

ABC is the required triangle.

For $\angle\ CAD$ = half of a right angle;

$\therefore CA = CD$, and $CB - CA = BD$.

37. When the given angle is acute:

Let AB be the side adjacent to the acute angle. At B make $\angle\ ABD$ = the given acute angle. With A as centre and radius = the side opposite the acute angle, describe a circle cutting BD at C; join AC. ABC is the required triangle.

The side opposite the acute angle B may be greater than AB, equal to AB, or less than AB.

When it is greater than AB, the circle described will cut BD not only at C, but at another point C', on the opposite side of B from C. If AC' be joined, it will be seen that $\triangle\ ABC'$ does not fulfil the given conditions; hence there is only one solution, namely $\triangle\ ABC$.

When it is equal to AB, the circle described will cut BD not only at C, but at B; hence there is only one solution, namely $\triangle\ ABC$.

When it is less than AB, the circle described may cut BD not only at C, but at another point C', on the same side of B as C. If AC' be joined, it will be seen that $\triangle\ ABC'$ does fulfil the given conditions; hence there are two solutions, namely $\triangle$s ABC, ABC'. If the circle described should meet BD at only one point C, there can be only one solution. If the circle described should not meet BD at all, the problem is impossible.

When the given angle is right:

Let AB be the side adjacent to the right angle. At B make $\angle\ ABD$ right. With A as centre and radius = the side opposite the right angle describe a circle cutting BD at C; join AC. ABC is the required triangle.

The side opposite the right angle B must, if the problem be possible, be greater than AB; the circle described will consequently cut BD not only at C, but at another point C' on the opposite side of B from C. If AC' be joined, it will be seen that $\triangle\ ABC'$ does fulfil the given conditions; hence there will be two solutions, namely $\triangle$s ABC, ABC'. As these triangles are congruent, it is usual to say there is only one solution.

When the given angle is obtuse:

Let AB be the side adjacent to the obtuse angle. At B make $\angle\ ABD$ = the given obtuse angle. With A as centre and radius = the side opposite the obtuse angle, describe a circle cutting BD at C; join AC.

ABC is the required triangle.

The side opposite the obtuse angle B must, if the problem be possible, be greater than AB; the circle described will consequently cut BD not only at C, but at another point C' on the opposite side of B from C. If AC' be joined, it will be seen that $\triangle\ ABC'$ does not fulfil the given conditions; hence there is only one solution, namely triangle ABC.

38. Let BC be the given side.

At B make $\angle\ CBD =$ the given angle,
and cut off $BD =$ the sum of the other two sides.
Join DC, at C make $\angle\ DCA = \angle\ BDC$,
and let CA meet BD at A. ABC is the required triangle.

For $AC = AD$ (I. 6); $\therefore\ BA + AC = BD$.

39. Let BC be the given side.

At B make $\angle\ CBD =$ the given angle,
and cut off $BD =$ the difference of the other two sides.
Join DC, at C make $\angle\ DCA =$ the supplement of $\angle\ BDC$,
and let CA meet BD produced at A.

ABC is the required triangle.

For $AC = AD$ (I. 6); $\therefore\ BA - AC = BD$.

40. Let BD be the sum of the two sides.

At D make $\angle\ BDC =$ half of the given angle;
with B as centre and radius $=$ the given side, describe a circle cutting DC at C.
At C make $\angle\ DCA = \angle\ BDC$,
and let CA meet BD at A. Join BC.

ABC is the required triangle.

For $\angle\ BAC =$ twice $\angle\ ADC =$ the given angle;
and $BA + AC = BD$.

The circle will in general cut DC at another point C'. If at C', $\angle\ DC'A'$ is made $= \angle\ BDC$, $C'A'$ meeting BD at A', then $\triangle\ A'BC'$ will satisfy the given conditions. It may be proved, however, that $\triangle$s ABC and $A'BC'$ are congruent.

41. Let BD be the difference of the two sides.

At D make $\angle\ BDC =$ a right angle $+$ half of the given angle;
with B as centre, and radius $=$ the given side, describe a circle cutting DC at C.
At C make $\angle\ DCA =$ the supplement of $\angle\ BDC$,
and let CA meet BD produced at A. Join BC.

ABC is the required triangle.

For $\angle\ BAC = \angle\ BDC - \angle\ DCA$, *I.* 32

$= \angle\ BDC - (2 \text{ rt. } \angle\text{s} - \angle\ BDC)$,

$= 2\ \angle\ BDC - 2 \text{ rt. } \angle\text{s}$,

$= 2\,(1 \text{ rt. } \angle + \frac{1}{2} \text{ given angle}) - 2 \text{ rt. } \angle\text{s}$,

$=$ given angle;

and $BA - AC = BD$.

42. Construct, by the thirtieth deduction, a right-angled triangle

ADE whose hypotenuse AE = the given bisector, and whose side AD = the given perpendicular.
At A and on opposite sides of AE make $\angle$s EAB, EAC, each = half of the given angle,
and produce DE to meet AB and AC at B and C.

ABC is the required triangle.

43. Let BD be the sum of the two sides.
At D make $\angle\ BDC$ = half of the first given angle,
at B make $\angle\ DBC$ = the second given angle, and let BC meet DC at C.
At C make $\angle\ DCA = \angle\ BDC$, and let CA meet BD at A.

ABC is the required triangle.

For $\angle\ BAC$ = twice $\angle\ ADC$ = the first angle;
$\angle\ ABC$ = the second angle; $\therefore$ $\angle\ ACB$ = the third angle;
and $BA + AC = BD$.

44. Let BD be the difference of the two sides.
At D make $\angle\ BDC$ = a right angle + half of the first given angle,
at B make $\angle\ DBC$ = the second given angle, and let BC meet DC at C.
At C make $\angle\ DCA$ = the supplement of $\angle\ BDC$,
and let CA meet BD produced at A.

ABC is the required triangle.

For $\angle\ BAC = \angle\ BDC - \angle\ DCA$, *I.* 32
$= \angle\ BDC -$ (2 rt. $\angle$s $- \angle\ BDC$),
$= 2\ \angle\ BDC -$ 2 rt. $\angle$s,
$= 2$ (1 rt. $\angle$ + $\frac{1}{2}$ first angle) $-$ 2 rt. $\angle$s,
= the first angle;
$\angle\ ABC$ = the second angle; $\therefore$ $\angle\ ACB$ = the third angle;
and $BA - AC = BD$.

45. Let DE be the perimeter.
At D and E make $\angle$s EDA, DEA respectively equal to half the base angles, and let DA, EA meet at A.
At A make $\angle\ DAB = \angle\ D$, and $\angle\ EAC = \angle\ E$,
and let AB, AC meet DE at B and C.

ABC is the required triangle.

For $\angle\ ABC$ = twice $\angle\ D$, and $\angle\ ACB$ = twice $\angle\ E$; *I.* 32
and $AB = DB$, and $AC = EC$.

46. Case first: When two sides and the median to the third side are given, as AB, AC, AH in the fig. on p. 100 of *Euclid*.
Construct a $\triangle\ ABA'$ such that AB = one of the sides,

BA' = the other side, and AA' = twice the median. Bisect AA' at H; join BH and produce it to C, so that $HC = BH$; join AC. ABC is the required triangle.

For $\triangle$s AHC, $A'HB$ are congruent; $\therefore AC = A'B$. *I.* 4

Case second: When two sides and the median to one of them are given, as AB, BC, AH in the fig. on p. 100 of *Euclid*.

Construct a $\triangle$ ABH such that AB = one of the sides, BH = half of the other side, and AH = the median. Produce BH to C, so that $HC = BH$, and join AC.

ABC is the required triangle.

47. Let BK, CL, in the fig. on p. 100 of *Euclid*, be the given medians, BC the given side.

Construct a $\triangle$ BCG having BC = the given side, and BG, CG respectively two-thirds of the given medians.

Produce BG to K and CG to L, so that GK = half of BG, and GL = half of CG.

Join BL, CK and produce them to meet at A.

ABC is the required triangle.

Let BK, CL be the given medians, AB the given side.

Construct a $\triangle$ BGL having BG = two-thirds of the one median, GL = one-third of the other median, and LB = half of the given side.

Produce BL to A so that $LA = LB$, and LG to C so that GC = twice LG; join AC, BC.

ABC is the required triangle.

The proof follows from App. I. 4.

48. Construct a $\triangle$ GBC such that GB = two-thirds of the second median, GC = two-thirds of the third median, and GH (the median of $\triangle$ GBC drawn to BC) = one-third of the first median; which may be done by the first case of the forty-sixth deduction.

Produce BG to K and CG to L, so that GK = half of BG, and GL = half of CG.

Join BL, CK and produce them to meet at A.

ABC is the required triangle.

49. Join BC in the fig. on p. 89 of *Euclid*.

Then the square $ABDC$ could be constructed, if $\triangle$ ABC could be constructed. Now in $\triangle$ ABC the angles are known and the sum of two sides; hence, by the application of the forty-third deduction, the square can be constructed.

50. Apply the forty-fourth deduction.

51. In the fig. to Def. 1 on p. 112 of *Euclid*, join AC, BD intersecting at E.
Then AC, BD are equal, and bisect each other.
Hence $\triangle\ EAB$ is isosceles, and if it could be constructed, so also could the rectangle.
Now in $\triangle\ EAB$ the vertical angle AEB is known, and also the base AB; at A and B, therefore, make $\angle$s BAE, ABE each = half the supplement of $\angle\ AEB$.

52. In the fig. to Def. 1 on p. 112 of *Euclid*, join AC.
Then the rectangle $ABCD$ could be constructed, if $\triangle\ ABC$ could be constructed. Now in $\triangle\ ABC$, AC is known, the angle opposite to it (a right angle) is known, and the sum of AB and BC (half the perimeter) is known; the solution therefore follows by the application of the fortieth deduction.

53. In the fig. to Def. 1 on p. 112 of *Euclid*, join AC, BD intersecting at E.
Then the rectangle $ABCD$ could be constructed if $\triangle\ ABC$ could be constructed.
Now since $\triangle\ EAB$ is isosceles, $\angle\ BAC$ = half the supplement of $\angle\ AEB$, which is known; and $\angle\ ABC$ is right.
Hence in $\triangle\ ABC$ the angles are known and the sum of AB and BC (half the perimeter); the solution therefore follows by the application of the forty-third deduction.

54. As in the previous deduction, the angles of $\triangle\ ABC$ are known, and the difference of AB and BC; the solution therefore follows by the application of the forty-fourth deduction.

55. In the fig. on p. 73 of *Euclid*, join AD intersecting BC at E.
Then AD, BC bisect each other; hence $\|^m\ ABDC$ could be constructed if $\triangle\ AEB$ could be constructed. Now in $\triangle\ AEB$ the three sides are known; the solution therefore follows by the application of I. 22.

56. The solution follows by the application of the eleventh deduction from I. 23, since in $\triangle\ AEB$, AE, BE, and $\angle\ AEB$ are known.

57. When one angle of a $\|^m$ is known, the others are also known, since they are either equal or supplementary to it; hence in $\triangle\ ABC$ (fig. on p. 73 of *Euclid*), AB, BC and $\angle\ A$ are known.
By application of the thirty-seventh deduction, therefore, $\triangle\ ABC$ may be constructed, and thence $\|^m\ ABDC$.

58. Let ABC (fig. on p. 84 of *Euclid*) be the given triangle.
Through A draw $AG \parallel BC$, and bisect BC in E. With E as

centre, and a radius = half of $(AB + AC)$ describe a circle cutting AG at F; join EF, and draw $CG \parallel EF$.

Then $\parallel^m$ $FECG = \triangle\ ABC$; *I.* 42

$EC + FG = 2\,EC = BC$; and $EF + CG = 2\,EF = AB + AC$.

59. Let $\parallel^m$ $EFGH$ be inscribed in $\parallel^m$ $ABCD$, E, F, G, H being situated respectively on AB, BC, CD, DA; join AC, EG intersecting at O.

Since the sides of $\triangle\ AEH$ are respectively $\parallel$ the sides of $\triangle\ CGF$, and are drawn in opposite directions;

$\therefore$ $\triangle\ AEH$ is equiangular to $\triangle\ CGF$, by the fourth deduction from I. 29.

Now $EH = GF$ (I. 34); $\therefore AE = CG$. *I.* 26

Hence $\triangle$s AOE, COG are congruent; *I.* 29, 26

$\therefore AO = CO$, and $EO = GO$, that is AC and EG intersect at their middle points.

But the diagonals of $\parallel^m$ $ABCD$ intersect at the middle point of AC,

and the diagonals of $\parallel^m$ $EFGH$ at the middle point of EG;

$\therefore$ the diagonals of all the $\parallel^{ms}$ inscribed in $\parallel^m$ $ABCD$ intersect at the point where the diagonals of $\parallel^m$ $ABCD$ intersect.

60. Let $ABCD$ be the given rhombus.

Join AC, BD intersecting at O;

bisect the four angles at O by the two straight lines EG and FH, E and G being situated on AB and CD, F and H on BC and DA. Join EF, FG, GH, HE.

$EFGH$ is the required square.

$ABCD$ is a $\parallel^m$, by the eighth deduction from I. 27;

$\therefore EO = GO$, $FO = HO$, by the twelfth deduction from I. 34;

$\therefore$ $EFGH$ is a $\parallel^m$, by the ninth deduction from I. 27;

$\therefore$ $EFGH$ is a rhombus, by the sixth deduction from I. 13, and the ninth deduction from I. 34.

Now by the third deduction from I. 8, and by I. 26, it may be proved that $\triangle$s AOE, AOH are congruent;

$\therefore$ $OE = OH$, and $EG = FH$;

$\therefore$ $EFGH$ is a square, by the tenth deduction from I. 34.

61. Let ABC be the given isosceles right-angled triangle, C being the right angle.

Trisect the hypotenuse AB at the points D, E; draw DG, $EF \perp AB$, and meeting AC, BC respectively in G, F; join FG. $DEFG$ is the required square.

Because $\angle\ A$ is half of a right angle, and $\angle\ ADG$ is right;

$\therefore$ $DG = DA$. Similarly $EF = BE$; $\therefore$ $DG = EF$.
Now DG and EF are parallel; $\therefore$ $DEFG$ is a $\|^m$;
$\therefore$ $DEFG$ is a square, by the first and third deductions from I. 34.

62. Let $ABDC$ (fig. on p. 89 of *Euclid*) be the given square.
At A make $\angle\ BAE$ and $\angle\ CAF$ each = one-sixth of a right angle, by the tenth deduction from I. 32, and by I. 9; let AE meet BD at E, and AF meet CD at F; join EF.

AEF is the required triangle.

For $\triangle$s ABE, ACF are congruent; *I.* 26
$\therefore$ $\triangle\ AEF$ is isosceles.
Now $\angle\ EAF = 1$ rt. $\angle - \frac{2}{6}$ rt. $\angle = \frac{2}{3}$ rt. $\angle$;
$\therefore$ $\triangle\ AEF$ is equilateral, by the twelfth deduction from I. 32.

63. Join PB, PC, PB', PC', RB, RB'.
Then by I. 38

$$\left.\begin{aligned}\triangle\ A'OB' &= \triangle\ APB' \\ \triangle\ A'OC' &= \triangle\ APC' \\ \triangle\ B'OC' &= \triangle\ B'RC\end{aligned}\right\}(1)$$

$$\left.\begin{aligned}\triangle\ A'OC &= \triangle\ APC,\ \triangle\ B'OP = \triangle\ BQP \\ \triangle\ C'OP &= \triangle\ CRP,\ \triangle\ BOC' = \triangle\ BRC \\ \triangle\ A'OB &= \triangle\ APB,\ \triangle\ B'OR = \triangle\ BQR\end{aligned}\right\}(2)$$

Hence from (1) by addition,

$$\triangle\ A'OB' + \triangle\ A'OC' + \triangle\ B'OC' = \triangle\ APB' + \triangle\ APC' + \triangle\ B'RC;$$

$$\begin{aligned}\therefore\ \triangle\ A'B'C' &= \triangle\ ABC - \text{polygon } PC'BCRB', \\ &= \triangle\ ABC - (A'OC + C'OP + A'OB + B'OP \\ &\qquad + BOC' + B'OR), \\ &= \triangle\ ABC - (APC + CRP + APB + BQP \\ &\qquad + BRC + BQR), \text{ by (2)} \\ &= \triangle\ PQR.\end{aligned}$$

[This solution is due to William Hopps, Hull.]

64. Produce HA to meet BC in N.
Because AB is $\|$ HL, and AH is $\|$ BL,
$\therefore$ $ABLH$ is a $\|^m$, and $BL = AH$.
Similarly $ACMH$ is a $\|^m$, and $CM = AH$;
$\therefore$ $BL = CM$, and $BCML$ is a $\|^m$.
Because $\|^m\ ABDE = \|^m\ ABLH$, *I.* 35
and $\|^m\ ABLH = \|^m\ LN$; *I.* 35
$\therefore$ $\|^m\ ABDE = \|^m\ LN$.
Similarly $\|^m\ ACFG = \|^m\ MN$;
$\therefore$ $\|^m\ BCML = \|^m\ ABDE + \|^m\ ACFG$.

65. In the fig. to the preceding deduction, make $\angle\ BAC$ right; on AB and AC describe squares; and complete the rest of the construction.

Then BA, AG can be proved to be in the same straight line, as well as CA and AE, as in I. 47; $\therefore$ $AEHG$ is a $\|^m$.
Hence $\triangle\ AEH$ can be proved congruent to $\triangle\ BAC$; *I.* 4
$\therefore$ $AH = BC$, and $\angle\ AHE = \angle\ BCA$.
Now $\angle\ EAH = \angle\ NAC$ (I. 15); $\therefore$ $\angle\ ANC = \angle\ AEH$;
$\therefore$ HN is $\perp$ BC, and $\|^m$ $BCML$ is a rectangle.
Also $BL = AH = BC$;
$\therefore$ $\|^m$ $BCML$ is a square.
The equality of $BCML$ to the sum of $ABDE$ and $ACFG$ is established as in the preceding deduction.

66. Let ABC be a triangle, and let the three concurrent straight lines OD, OE, OF be respectively $\perp$ BC, CA, AB or these sides produced: it is required to prove
$AF^2 + BD^2 + CE^2 = FB^2 + DC^2 + EA^2$.

Join O with the vertices A, B, C.

Then $AF^2 + BD^2 + CE^2 = (AO^2 - OF^2) + (BO^2 - OD^2) + (CO^2 - OE^2)$, *I.* 47, *Cor.*

and $FB^2 + DC^2 + EA^2 = (BO^2 - OF^2) + (CO^2 - OD^2) + (AO^2 - OE^2)$; *I.* 47, *Cor.*

$\therefore$ $AF^2 + BD^2 + CE^2 = FB^2 + DC^2 + EA^2$.

This result is sometimes written—
$(AF^2 - FB^2) + (BD^2 - DC^2) + (CE^2 - EA^2) = 0$.

Conversely:

If the sides of a triangle be divided so that the sums of the squares of the alternate segments taken cyclically are equal, the perpendiculars to the sides from the points of division are concurrent.

Let ABC be a triangle, and let its sides or its sides produced be divided at F, D, E so that $AF^2 + BD^2 + CE^2 = FB^2 + DC^2 + EA^2$:
it is required to prove that the perpendiculars to AB, BC, CA from the points F, D, E are concurrent.

At E and F draw EO, FO $\perp$ CA, AB, and let them meet each other at O;
from O draw OD' $\perp$ BC or BC produced.

Then $AF^2 + BD'^2 + CE^2 = FB^2 + D'C^2 + EA^2$.
But $AF^2 + BD^2 + CE^2 = FB^2 + DC^2 + EA^2$;

$\therefore BD^2 - BD'^2 = DC^2 - D'C^2$, by subtraction.
But BD^2 is greater than BD'^2, since BD is greater than BD';
$\therefore DC^2$ is greater than $D'C^2$, which is impossible.
Hence the perpendicular from O to BC must be OD.

67. APPENDIX I. 2.

Let H, K, L be the middle points of the sides BC, CA, AB of $\triangle ABC$.
Then $AL = LB$, $BH = HC$, $CK = KA$;
$\therefore AL^2 + BH^2 + CK^2 = LB^2 + HC^2 + KA^2$;
$\therefore$ the perpendiculars to BC, CA, AB at H, K, L are concurrent.

APPENDIX I. 3.

$$AZ^2 - ZB^2 = (AZ^2 + ZC^2) - (ZB^2 + ZC^2) = AC^2 - BC^2.$$
Similarly $BX^2 - XC^2 \qquad = AB^2 - AC^2$,
and $\qquad CY^2 - YA^2 \qquad = BC^2 - AB^2$;
$\therefore (AZ^2 - ZB^2) + (BX^2 - XC^2) + (CY^2 - YA^2) = 0$;
$\therefore AX, BY, CZ$ are concurrent.

68. Let ABC be a triangle (for convenience let $\angle ACB$ be obtuse), and let BA be produced to B'; let AN, AN' be the bisectors of $\angle BAC$ and $\angle B'AC$; from H, the middle point of BC, let there be drawn HD, HD' respectively $\perp AN, AN'$ produced, and meeting the sides AB, AC, or those sides produced in E, F, E', F': it is required to prove

$$\tfrac{1}{2}(AB + AC) = AE = AF = BE' = CF'.$$
$$\tfrac{1}{2}(AB - AC) = BE = CF = AE' = AF'.$$

Through C draw $CGG' \parallel AB$, and meeting the perpendiculars HD, HD' produced at G, G' respectively.

Because $\angle B'AN' = \angle AEF$, and $\angle CAN' = \angle AFE$, *I.* 29
$\therefore \angle AEF = \angle AFE$, and $AE = AF$. *I.* 6
Because CG is $\parallel AE$, $\therefore \angle CGF = \angle AEF = \angle CFG$;
$\therefore CF = CG$.
But since $\triangle$s BEH, CGH are congruent, *I.* 26
$\therefore BE = CG = CF$.
Now $\tfrac{1}{2}(AB + AC) = \tfrac{1}{2}\left\{(AE + BE) + (AF - CF)\right\}$,
$$= \tfrac{1}{2}(AE + AF),$$
$$= AE = AF.$$
Because $\angle B'AN' = \angle E'AD'$, and $\angle CAN' = \angle F'AD'$,
$\therefore \angle E'AD' = \angle F'AD'$;
and since the angles at D' are right, $\therefore AE' = AF'$. *I.* 26

Because $\angle\, CF'G' = \angle\, AE'F' = \angle\, CG'F'$, *I.* 29

$\therefore\ CF' = CG'$.

But since $\triangle$s $BE'H$, $CG'H$ are congruent, *I.* 26

$\therefore\ BE' = CG' = CF'$.

Now $\frac{1}{2}(AB + AC) = \frac{1}{2}\{(BE' + AE') + (CF' - AF')\}$,

$= \frac{1}{2}\ (BE' + CF')$,

$= BE' = CF'$.

Lastly $\frac{1}{2}(AB - AC) = \frac{1}{2}\{(AE + BE) - (AF - CF)\}$,

$= \frac{1}{2}\ (BE + CF)$,

$= BE = CF$;

and $\frac{1}{2}(AB - AC) = \frac{1}{2}\{(BE' + AE') - (CF' - AF')\}$,

$= \frac{1}{2}\ (AE' + AF')$

$= AE' = AF'$.

[Another proof of this theorem will be found in M'Dowell's *Exercises on Euclid and in Modern Geometry*, section 112.]

69. $\frac{1}{2}(\angle\, C + \angle\, B) = \angle\, B'AN' = \angle\, CAN' = \angle\, E'AD' = \angle\, F'AD'$
$= \angle\, AEF = \angle\, AFE = \angle\, CGF = \angle\, HGG'$;

$\frac{1}{2}(\angle\, C - \angle\, B) = \angle\, BHE = \angle\, CHG = \angle\, AN'C$.

For $\angle\, C + \angle\, B = \angle\, B'AC$; *I.* 32

$\therefore\ \frac{1}{2}(\angle\, C + \angle\, B) = \angle\, B'AN' = CAN'$.

The other equalities follow by I. 15, 29.

$\frac{1}{2}(\angle\, C - \angle\, B) = \frac{1}{2}(\angle\, C + \angle\, B) - \angle\, B$,

$= \angle\, AEF - \angle\, B$,

$= \angle\, BHE$. *I.* 32

The other equalities follow by *I.* 15, 29.

70. Let ABC be a triangle, and let BD which bisects $\angle\, ABC$, and CE which bisects $\angle\, ACB$, be equal; to prove $\angle\, ABC = \angle\, ACB$.

Through D draw $DF \parallel BA$, and through E draw $EF \parallel BD$ and join CF.

If $\angle\, ABC$ be not $= \angle\, ACB$, let $\angle\, ABC$ be the greater; then $\angle\, DBC$ is greater than $\angle\, ECB$.

In $\triangle$s DBC, ECB, $\left\{\begin{array}{l} DB = EC \quad \textit{Hyp.} \\ BC = CB \\ \angle\, DBC \text{ is greater than } \angle\, ECB; \end{array}\right.$

$\therefore\ DC$ is greater than EB. *I.* 24

Now $EB = DF$, because $BDFE$ is a $\parallel^{m}$;

$\therefore\ DC$ is greater than DF;

$\therefore\ \angle\, DFC$ is greater than $\angle\, DCF$.

Again $EC = BD$ (hyp.) $= EF$; *I.* 34

$\therefore \angle ECF = \angle EFC$; *I.* 5

$\therefore \angle ECF - \angle DCF$ is greater than $\angle EFC - \angle DFC$;

$\therefore \angle ECD$ is greater than $\angle EFD$;

$\therefore \angle ECD$ is greater than $\angle EBD$; *I.* 34

$\therefore \angle ACB$ is greater than $\angle ABC$, which is absurd.

Hence $\angle ABC = \angle ACB$, and $\triangle ABC$ is isosceles.

[This proof is given by M. Descube in the *Journal de Mathématiques Élémentaires et Spéciales*, 1880, vol. iv., p. 538. For a direct proof see the last reference given in the *Euclid.*]

If the straight lines bisecting the angles below the base and terminated by the opposite sides be equal, the triangle is isosceles. The construction and proof of this case are closely analogous to those of the preceding, almost the only changes being that the pair of angles which were there subtracted are here added, and that subsequently 'greater than' has to be replaced by 'less than.'

LOCI.

1. Let AB be the given straight line, M the given distance.

   Take any point C in AB; draw $CD \perp AB$, and $= M$; through D draw $EF \parallel AB$.

   Then it may be proved by I. 28, 34, that the distance of any point in EF from $AB = CD = M$.

   If a similar construction be made on the opposite side of AB from EF, the distance of every point in this second parallel from AB will $= M$.

2. Let BCD be the $\bigcirc^{ce}$ of the given circle, A its centre, and M the given distance.

   Draw any radius AB; produce AB to E, so that $BE = M$, and from BA cut off $BF = M$. With centre A, and radii AE, AF, describe two circles EGH, FKL.

   Then if any other radius be drawn to meet the $\bigcirc^{ces}$ BCD, EGH, FKL, it may be shown that the part of it intercepted between BCD and EGH, or between BCD and FKL, $= BE$ or $BF = M$.

   If $M = AB$, the $\bigcirc^{ce}$ FKL reduces to a point;

   if M be greater than AB, the $\bigcirc^{ce}$ FKL becomes impossible.

3. Let AB and CD (fig. to I. 15) intersect at E, and let FEG bisect $\angle$s AED, BEC, and HEK bisect $\angle$s BED, AEC.

Take any point P in EH, and draw $PM \perp AB$, and $PN \perp CD$.

Then $\triangle$s PEM, PEN are congruent; *I.* 26

$\therefore PM = PN$.

Similarly, any point in EF, EG, EK is equidistant from AB and CD.

4. When AB and CD are parallel, the locus is a straight line parallel to AB and CD, drawn through the middle point of any straight line which joins AB to CD.

5. Let BC be the given base, M the given length.

With B as centre and M as radius describe a circle. Then if any point A on the $\bigcirc^{ce}$ of this circle be joined to B and C, $\triangle\ ABC$ will be on the given base and have one of its sides = the given length.

Similarly, if any point on the $\bigcirc^{ce}$ of the circle described with C as centre and M as radius, be joined to B and C, a triangle will be obtained fulfilling the given conditions.

6. Let BC be the given base, $\angle\ M$ the given angle.

At B and on opposite sides of BC make $\angle$s CBA, CBA', each $= \angle\ M$; then if any point be taken in BA or BA' and joined to B and C, a triangle will be obtained fulfilling the given conditions.

Similarly, if at C, and on opposite sides of BC, $\angle$s BCA, BCA' be made each $= \angle\ M$, any point in CA or CA' will be the vertex of a triangle fulfilling the given conditions.

Let BA, CA meet at D and BA', CA' at D'; then $DBD'C$ is a rhombus.

When $\angle\ M$ is acute, the sides as well as the sides produced of the rhombus $DBD'C$ constitute the locus; when $\angle\ M$ is obtuse, the sides produced alone constitute the locus; when $\angle\ M$ is right, the locus consists of the two perpendiculars to BC at the point B and C.

7. Let A be the given point, M the given straight line.

With A as centre and M as radius describe a circle.

The $\bigcirc^{ce}$ of this circle is the required locus.

8. Let B and C be the given points.

Join BC; bisect it at D, and through D draw a perpendicular to BC. Any point in this perpendicular is equidistant from B and C, and consequently the centre of a circle which

passes through B and C. This perpendicular, therefore, is the required locus.

9. This is merely the previous question expressed differently.

10. The locus is the ○^{ce} of the circle described with the middle point of the base as centre and with the median as radius.

11. The locus consists of two parallels to the base, equidistant from it, and on opposite sides of it.

12. The locus is the same as in the preceding question, for if the triangles have equal areas, they will have equal altitudes.

13. Let A be the given point, BC the given straight line, and let AD, AE be two of the straight lines drawn from A to BC.

    Then the straight line joining the middle points of AD and AE is parallel to DE. *App. I.* 1

    Hence the locus is the straight line parallel to BC, and equidistant from A and BC.

14. By App. I. 1 and Cor. 1 it will be seen that the locus is a straight line parallel to the parallels and equidistant from them.

15, 16. Since the diagonals of the ||^{ms} bisect each other, the locus of their intersection is the same as in 13.

17. By the sixth deduction from I. 40, the median from the vertex to the base bisects every parallel to the base; hence the locus is this median produced indefinitely either way.

18. The second deduction from I. 39 proves that the distance of the vertex of the right angle in a right-angled triangle from the middle point of the hypotenuse = half the hypotenuse; hence the required locus is the ○^{ce} of the circle described with the middle point of the hypotenuse as centre and half the hypotenuse as radius.

19. The second deduction from I. 39 proves that the distance of the middle point of the ladder from the foot of the perpendicular wall is always equal to half the length of the ladder; hence the middle point of the ladder describes the fourth part of the ○^{ce} of the circle whose centre is the foot of the wall and whose radius is half the length of the ladder.

20. Let AB, CD be two equal segments of the straight line.

    Bisect BC in E, and from E draw a perpendicular to BC. This perpendicular is the locus.

    For if any point P be taken in this perpendicular, and joined to A, B, C, D, it may be proved that

    $\angle APE = \angle DPE$, and $\angle BPE = \angle CPE$; *I.* 4

$\therefore \angle APE - \angle BPE = \angle DPE - \angle CPE;$

$\therefore \angle APB = \angle DPC.$

21. It is easy to see that the locus is the $\bigcirc^{ce}$ of a circle; but the following proof may be given.

Let AB (fig. on p. 72 of *Euclid*) be the given straight line, and D the centre of the circle on whose $\bigcirc^{ce}$ the extremity B moves, so that D is a fixed point. From D draw $DC \parallel BA$ and $= BA$; then C is a fixed point. Now let $A'B'$ be a second position of AB; $A'B'$ is $\parallel AB$ and $= AB$, and B' is on the $\bigcirc^{ce}$ of the circle whose centre is D. *Hyp.*

Join CA, CA', DB, DB'.

Then $ABDC$ is a $\parallel^m$ (I. 33); $\therefore CA = DB$;

and $A'B'DC$ is a $\parallel^m$ (I. 33); $\therefore CA' = DB'$.

Now $DB = DB' =$ a fixed distance;

$\therefore CA = CA' =$ the same fixed distance;

$\therefore$ the extremity A when it moves is always at a fixed distance from the fixed point C.

Hence the locus of A is the $\bigcirc^{ce}$ of a circle, whose centre is C, and whose radius = the radius of the other circle.

22. Of all the triangles whose base is the given base BC, and whose median BK = the given length, let ABC be one.

Through A draw $AO \parallel KB$ meeting CB produced at O.

Then $OB = BC$ and $OA = 2\,BK$; *App. I.* 1, *Cor.* 1

$\therefore O$ is a fixed point, and OA a fixed distance.

Hence the vertex A is situated at a fixed distance (twice the given median) from a fixed point;

$\therefore$ the locus of A is the $\bigcirc^{ce}$ of the circle described with O as centre, and radius twice the median from B.

23. Of all the triangles whose base is the given base BC, and the difference of whose sides = the given difference, let ABC be one.

Bisect the interior vertical angle BAC by AD;

from B and C draw BD and $CE \perp AD$,

and let BD, CE meet the sides AC, AB in F and G:

it is required to find the locus of the points D and E.

By application of I. 26 it may be seen, on comparing $\triangle$s AEC, AEG, that $CE = EG$, and $AC = AG$;

and on comparing $\triangle$s ADB, ADF, that $BD = DF$, and $AB = AF$.

Hence the given difference of sides $AB - AC = AB - AG = BG$,

or $= AF - AC = CF$.

Take H the middle point of BC, and join HD, HE.
Then $HD = \frac{1}{2} CF$, and $HE = \frac{1}{2} BG$. *App. I.* 1
But because BC is a fixed straight line, H its middle point is a fixed point;
and it has been shown that the distances of the points D and E from the fixed point H are each = half the given difference of sides;
$\therefore$ the locus of D and E is the $\bigcirc^{ce}$ of a circle whose centre is H and whose radius = half the given difference of sides.

24. Of all the triangles whose base is the given base BC, and the sum of whose sides = the given sum, let ABC be one.
Bisect the exterior vertical angle BAF or CAG by DAE;
from B and C draw BD and $CE \perp DAE$,
and let BD, CE meet the sides AC, AB in F and G:
it is required to find the locus of the points D and E.

By application of I. 26 it may be seen, on comparing $\triangle$ AEC, AEG, that $CE = EG$, and $AC = AG$;
and on comparing $\triangle$s ADB, ADF, that $BD = DF$, and $AB = AF$.
Hence the given sum of sides $AB + AC = AB + AG = BG$,
or $= AF + AC = CF$.

Take H the middle point of BC, and join HD, HE.
Then $HD = \frac{1}{2} CF$, and $HE = \frac{1}{2} BG$. *App. I.* 1
But because BC is a fixed straight line, H its middle point is a fixed point;
and it has been shown that the distances of the points D and E from the fixed point H are each = half the given sum of sides;
$\therefore$ the locus of D and E is the $\bigcirc^{ce}$ of a circle whose centre is H, and whose radius = half the given sum of sides.

25. Let AB, BC, CD be the three given sides, AC the given diagonal of the quadrilateral.

(1) Since AB, BC, AC are given, $\triangle$ ABC can be determined. *I.* 22
But since in the $\triangle$ ACD only AC and CD are known,
the vertex D cannot be determined.
Its locus is the $\bigcirc^{ce}$ of a circle with centre C and radius CD.

(2) Let E be the middle point of BD.
Then since from a fixed point B a variable line BD is drawn to the $\bigcirc^{ce}$ of the circle whose centre is C and radius CD,

the locus of the middle point of BD is the $\bigcirc^{ce}$ of another circle. *App. I.* 6
The centre of that circle will be G, the middle point of BC, and its radius will be GE, which $= \frac{1}{2}\, CD$.

(3) Let F be the middle point of AC;
join EF and bisect it at H.
Then since from a fixed point F a variable line FE is drawn to the $\bigcirc^{ce}$ of the circle whose centre is G and radius GE, the locus of H, the middle point of FE, is the $\bigcirc^{ce}$ of another circle. *App. I.* 6
The centre of that circle will be K, the middle point of FG, and its radius will be KH, which $= \frac{1}{2}\, GE$, and $\therefore = \frac{1}{4}\, CD$.

BOOK II.

PROPOSITION 1.

1. Let AB and CD be two straight lines (fig. to II. 1);
and let $CD = 2\, CE$:
it is required to prove $AB \cdot CD = 2\, AB \cdot CE$.
From C draw $CG \perp CD$, and $= AB$; *I.* 11, 3
through G draw $GH \parallel CD$,
and through E and D draw EK, $DH \parallel CG$. *I.* 31
Because $CE = ED$, $\therefore\ CK = EH$. *I.* 36
Hence $CH = CK + EH,$
$= 2\, CK;$
$\therefore\ GC \cdot CD = 2\, GC \cdot CE;$
$\therefore\ AB \cdot CD = 2\, AB \cdot CE.$

2. Let AB and CD be two straight lines (fig. to II. 1);
and let $CE = EF = FD$:
it is required to prove $AB \cdot CD = 3\, AB \cdot CE$.
Make the same construction as in II. 1.
Because $CE = EF = FD$, $\therefore\ CK = EL = FH$. *I.* 36
Hence $CH = CK + EL + FH,$
$= 3\, CK;$
$\therefore\ GC \cdot CD = 3\, GC \cdot CE;$
$\therefore\ AB \cdot CD = 3\, AB \cdot CE.$

3. Let AB and CE be two equal straight lines (fig. to II. 1).

From C draw $CG \perp CE$, and $= AB$; *I.* 11, 3

through G draw $GK \parallel CE$, and through E draw $EK \parallel CG$.

Then $CK = AB \cdot CE$.

But since $AB = CE$, and $CG = AB$; *Hyp.*, *Const.*

$\therefore \quad CG = CE$.

Hence the rectangle CK has two conterminous sides equal;

$\therefore$ all its sides must be equal; *I.* 34

$\therefore$ it is a square;

$\therefore AB \cdot CE = CE^2$.

4. Let AB and CD be two straight lines, let AB be divided internally at G into any two segments, and let CD be divided internally at E and F into any three segments:

it is required to prove $AB \cdot CD = AG \cdot CE + AG \cdot EF + AG \cdot FD + GB \cdot CE + GB \cdot EF + GB \cdot FD$.

$AB \cdot CD = AB \cdot CE + AB \cdot EF + AB \cdot FD$. *II.* 1

But $AB \cdot CE = AG \cdot CE + GB \cdot CE$, *II.* 1

$AB \cdot EF = AG \cdot EF + GB \cdot EF$, *II.* 1

$AB \cdot FD = AG \cdot FD + GB \cdot FD$; *II.* 1

$\therefore \quad AB \cdot CD = AG \cdot CE + AG \cdot EF + AG \cdot FD + GB \cdot CE + GB \cdot EF + GB \cdot FD$.

PROPOSITION 2.

1. Take $GH = AB$. *I.* 3

Then $GH \cdot AB = GH \cdot AC + GH \cdot CB$; *II.* 1

$\therefore \quad AB^2 = AB \cdot AC + AB \cdot CB$.

2. Let AB be divided internally at C and D:

to prove $AB^2 = AB \cdot AC + AB \cdot CD + AB \cdot DB$.

Take another straight line $EF = AB$. *I.* 3

Then $EF \cdot AB = EF \cdot AC + EF \cdot CD + EF \cdot DB$; *II.* 1

$\therefore \quad AB^2 = AB \cdot AC + AB \cdot CD + AB \cdot DB$.

3. Let AB be divided internally at C, D, E:

it is required to prove

$AB^2 = AB \cdot AC + AB \cdot CD + AB \cdot DE + AB \cdot EB$.

Take another straight line $FG = AB$. *I.* 3

Then $FG \cdot AB = FG \cdot AC + FG \cdot CD + FG \cdot DE + FG \cdot EB$; *II.* 1

$\therefore AB^2 = AB \cdot AC + AB \cdot CD + AB \cdot DE + AB \cdot EB$.

4. Consider AC and CB as two straight lines;
then AB is the sum of the two straight lines,
and AB^2 is the square on the sum of the two.
Also $AB \cdot AC$ is the rectangle contained by the sum and one of the straight lines;
and $AB \cdot CB$ is the rectangle contained by the sum and the other of the straight lines.

5. Consider AB and AC as two straight lines, of which AB is the greater;
then CB is the difference between the two straight lines.
Now $AB^2 =$ the square on the greater of the two straight lines;
$AB \cdot AC =$ the rectangle contained by the two straight lines;
$AB \cdot CB =$ the rectangle contained by the greater and the difference between the two.

PROPOSITION 3.

1. Take $GH = AB$. *I.* 3
Then $GH \cdot AB + GH \cdot CB = GH \cdot AC$; *II.* 1
$\therefore \quad GH \cdot AB = GH \cdot AC - GH \cdot CB$;
$\therefore \quad AB^2 = AB \cdot AC - AB \cdot CB$.

2. Consider AB and BC as two straight lines;
then AC is the sum of the two straight lines;
and $AB \cdot AC =$ the rectangle contained by the sum of the two straight lines and one of them.
Also $AB^2 =$ the square on that one of the straight lines;
and $AB \cdot CB =$ the rectangle contained by the two straight lines.
Now $AB \cdot AC = AB^2 + AB \cdot CB$. *II.* 3

3. Consider AC and AB as two straight lines, of which AB is the less;
then CB is the difference of the two straight lines.
Now $AB \cdot AC =$ the rectangle contained by the two straight lines;
$AB^2 =$ the square on the less;
$AB \cdot CB =$ the rectangle contained by the less and the difference of the two straight lines;
and $AB \cdot AC = AB^2 + AB \cdot CB$. *II.* 3

PROPOSITION 4.

1. HF and CK.
2. $ADEB$.
3. Gnomon AKF.
4. Gnomon AFK.
5. CK.
6. HF.
7. $AG + GE$, or $2\,AC \cdot CB$.
8. Consider AC and CB as two straight lines;
   then AB is the sum of the two straight lines,
   and AB^2 is the square on the sum of the two straight lines.
   Also $AC^2 + CB^2$ is the sum of the squares on the two straight lines;
   and $2\,AC \cdot CB$ is twice the rectangle contained by the two straight lines.
   Now AB^2 is greater than $AC^2 + CB^2$ by $2\,AC \cdot CB$. *II.* 4
9. Let AC and CB be equal;
   then $AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$, *II.* 4
   $= AC^2 + AC^2 + 2\,AC \cdot AC$,
   $= 4\,AC^2$.
10. Let AB be divided internally at C and D.
   On AB describe the square $AEFB$, and join BE. *I.* 46
   Through C and D draw CHK, $DLN \parallel AE$ or BF,
   meeting BE in H and L, and EF in K and N;
   through H and L draw $RHMS$, $PGLQ \parallel AB$ or EF,
   meeting AE, CK, DN, BF in R, H, M, S, and P, G, L, Q.
   Then it may be proved, as in II. 4, that RK, GM, DQ are the squares on AC, CD, DB;
   that AG, MF are each $= AC \cdot DB$;
   that PH, HN are each $= AC \cdot CD$;
   that CL, LS are each $= CD \cdot DB$;
   and that these nine figures make up $AEFB$.
   Hence $AB^2 = AC^2 + CD^2 + DB^2 + 2AC \cdot DB + 2\,AC \cdot CD + 2\,CD \cdot DB$.
11. Let $AC = a$, $CD = b$, $DB = c$; then $AB = a + b + c$.
   Now $AB^2 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,bc + 2\,ca$;
   and $AC^2 + CD^2 + DB^2 + 2\,AC \cdot DB + 2\,AC \cdot CD + 2\,CD \cdot DB$,
   $= a^2 + b^2 + c^2 + 2\,ac + 2\,ab + 2\,bc$.

PROPOSITION 5.

1. By CD^2. For $CB^2 - CD^2 = AD \cdot DB$;
   that is, $AC \cdot CB - CD^2 = AD \cdot DB$.
   Hence, if D be any point in AB, the rectangle $AD \cdot DB$ is always less than the square on AC, the half of AB, by CD^2;
   $\therefore AD \cdot DB$ increases as CD^2 diminishes;
   $\therefore AD \cdot DB$ is greatest when CD^2 is least, that is, when the points D and C coincide.
2. $AD \cdot DB$ diminishes as CD^2 increases, that is, as the point D moves farther from C, the middle of AB.
3. $AC = \frac{1}{2} AB = \frac{1}{2}(AD + DB)$.
   Cut off from AC a part $AE = DB$.
   Since $AC = CB$ and $AE = DB$,
   $\therefore EC = CD$, and $CD = \frac{1}{2} ED = \frac{1}{2}(AD - AE)$
   $= \frac{1}{2}(AD - DB)$.
4. $CEGD$ and $LEFM$.
5. $CEFB$.
6. $LEGH$.
7. Consider AD and DB as two straight lines;
   then CB = half the sum of AD and DB,
   and CD = half the difference of AD and DB.
   Now $AD \cdot DB = CB^2 - CD^2$. *II.* 5
8. Perimeter of the rectangle $AD \cdot DB = 2(AD + DB)$
   $= 2\,AB = 4\,CB$ = perimeter of the square on CB.
9. CB^2 was shown to be greater than $AD \cdot DB$ by CD^2;
   and the perimeter of the square on CB was shown to be equal to the perimeter of the rectangle $AD \cdot DB$.
10. Construct a rectangle whose sides shall be the sum and the difference of the sides of the two given squares. *II.* 5, *Cor.*
11. $AD^2 + DB^2 + 2\,AD \cdot DB = AB^2$, a fixed magnitude; *II.* 4
    $\therefore AD^2 + DB^2$ will be least when $AD \cdot DB$ is greatest,
    that is, when D is the middle point of AB, by the first deduction from II. 5.
12. Let $\triangle ABC$ be right-angled at C;
    then $AC^2 = AB^2 - BC^2$, *I.* 47, *Cor.*
    $= (AB + BC) \cdot (AB - BC)$. *II.* 5, *Cor.*

PROPOSITION 6.

1. $AD \cdot DB$ may be less than, equal to, or greater than $AC \cdot CB$.
   For $AD \cdot DB = CD^2 - CB^2 = CD^2 - AC \cdot CB$; *II.* 6
   $\therefore\ AD \cdot DB - AC \cdot CB = CD^2 - 2\,AC \cdot CB,$
   $= CD^2 - 2\,CB^2.$
   Now, when D is near to B, CD^2 may be less than $2\,CB^2$; as D moves farther from B, CD^2 increases while $2\,CB^2$ always remains the same. Hence CD^2 will become $= 2\,CB^2$, and greater than $2\,CB^2$, as D continues its movement.
2. For $AD \cdot DB = CD^2 - CB^2$; *II.* 6
   and, as D moves farther from B, CD^2 increases, while CB^2 remains constant;
   $\therefore$ $AD \cdot DB$ increases as D moves farther from B.
3. $AC = \frac{1}{2} AB = \frac{1}{2} (AD - DB)$.
   Produce CA to E making $AE = DB$.
   Since $AC = CB$, and $AE = DB$,
   $\therefore\ EC = CD$, and $CD = \frac{1}{2} ED = \frac{1}{2} (AD + AE)$
   $= \frac{1}{2} (AD + DB)$.
4. $CEGD$ and $LEFM$.
5. $LEGH$.
6. $CEFB$.
7. Consider AD and DB as two straight lines;
   then CD = half the sum of AD and DB,
   and CB = half the difference of AD and DB.
   Now $AD \cdot DB = CD^2 - CB^2$. *II.* 6
8. Perimeter of the rectangle $AD \cdot DB = 2\,(AD + DB)$
   $= 4\,CD$ = perimeter of the square on CD.

PROPOSITION 7.

1. HF and CK.
2. $ADEB$.
3. Gnomon AKF.
4. CK.
5. $AC^2 + CB^2 = HF + CK$; and AB^2, the square on the difference of AC and $CB = ADEB$.
   Hence $AC^2 + CB^2$ exceeds AB^2 by the gnomon $AKF + CK$, that is, by $2\,AC \cdot CB$.

6. Consider AC and CB as two straight lines;
   then AB is the difference of the two straight lines,
   and AB^2 is the square on the difference of the two straight lines.
   Also $AC^2 + CB^2$ is the sum of the squares on the two straight lines;
   and $2\,AC \cdot CB$ is twice the rectangle contained by the two straight lines.
   Now AB^2 is less than $AC^2 + CB^2$ by $2\,AC \cdot CB$. *II.* 7
7. For $AC^2 + CB^2 - 2\,AC \cdot CB = AB^2$, *II.* 7
   that is, the sum of the squares on AC and CB is greater than twice the rectangle $AC \cdot CB$ by AB^2.
   When AB^2 vanishes, that is, when AC becomes $= CB$,
   the sum of the squares on AC and $CB = 2\,AC \cdot CB$.
8. Let AB be divided internally at C,
   and let $AC^2 + CB^2 = 2\,AC \cdot CB$.
   Then $AC^2 + CB^2 - 2\,AC \cdot CB = 0$;
   $\therefore (AC - CB)^2 = 0$;
   $\therefore\ AC - CB = 0$, that is, $AC = CB$.

PROPOSITION 8.

1. $AEDC$. 2. $KLMN$. 3. AK, CL, DM, EN.
4. The square on the sum of AB and BC is $AEDC$;
   and the sum of the squares on AB and BC is the figure $ABLMGE$.
   Hence the excess of the former above the latter $= CL + DM$ $= 2\,AB \cdot BC$.
5. The sum of the squares on AB and BC is the figure $ABLMGE$;
   and the square on the difference of AB and BC is $KLMN$.
   Hence the excess of the former above the latter $= AK + EN$ $= 2\,AB \cdot BC$.

PROPOSITION 9.

1. Compare the 'Otherwise,' for the first part.
   Again considering AD and DB as two straight lines,
   $AC = \frac{1}{2}(AD + DB)$, and $CD = \frac{1}{2}(AD - DB)$, by the third deduction from II. 5.
   Now $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$. *II.* 9

2. By $2\,CD^2$. For $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2,$
$$= AC^2 + CB^2 + 2\,CD^2.$$

3. If D be any point in AB, $AD^2 + DB^2$ is greater than $AC^2 + CB^2$, a fixed magnitude, by $2\,CD^2$;
   $\therefore$ $AD^2 + DB^2$ diminishes as $2\,CD^2$ diminishes,
   that is, as D moves nearer to C.
   Hence $AD^2 + DB^2$ is least when D and C coincide.

4. $AD^2 + DB^2$ increases as $2\,CD^2$ increases,
   that is, as D moves nearer to either end of AB.

5. $4\,CD^2 + 2\,AD \cdot DB = 4\,CD^2 + 2\,(CB^2 - CD^2),$ *II. 5*
$$= 4\,CD^2 + 2\,(AC^2 - CD^2),$$
$$= 2\,AC^2 + 2\,CD^2,$$
$$= AD^2 + DB^2.$$ *II. 9*

6. Let EAB be an isosceles triangle, having $\angle\ E$ right, and let D, any point in the base AB, be joined to E;
   it is required to prove $2\,ED^2 = AD^2 + DB^2$.
   Draw $EC \perp AB$. *I. 12*
   Then since $\angle\ A$ is half a right angle, and $\angle\ ACE$ is right;
   $\therefore$ $\angle\ CEA$ = half a right angle; *I. 32*
   $\therefore$ $AC = CE$. *I. 6*
   Also $AC = CB$, by the third deduction from I. 26.
   Hence $2\,ED^2 = 2\,EC^2 + 2\,CD^2,$ *I. 47*
$$= 2\,AC^2 + 2\,CD^2,$$
$$= AD^2 + DB^2.$$ *II. 9*

PROPOSITION 10.

1. Compare the 'Otherwise' for the first part.
   Again, considering AD and DB as two straight lines, $CD = \frac{1}{2}(AD + DB)$, and $AC = \frac{1}{2}(AD - DB)$, by the third deduction from II. 6.
   Now $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$. *II. 10*

2. By $2\,CD^2$. For $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2,$ *II. 10*
$$= AC^2 + CB^2 + 2\,CD^2.$$

3. If D be any point in AB produced, $AD^2 + DB^2$ is greater than $AC^2 + CB^2$, a fixed magnitude, by $2\,CD^2$;
   $\therefore$ $AD^2 + DB^2$ diminishes as $2\,CD^2$ diminishes,
   that is, as D moves nearer to C.

4. $4\,CD^2 - 2\,AD \cdot DB = 4\,CD^2 - 2\,(CD^2 - CB^2)$, *II. 6*
$= 4\,CD^2 - 2\,(CD^2 - AC^2)$,
$= 2\,AC^2 + 2\,CD^2$,
$= \;\; AD^2 + \;DB^2$. *II. 10*

5. The proof of this is identical with the proof of the sixth deduction from II. 9, the only alteration required being the addition of the word 'produced.'

PROPOSITION 11.

1. For the angle supplementary to $\angle\ BAC$ is right; *I. 13*
and $\angle\ FAH$ of the square on AF is right.
Hence AH must coincide with AB.

2. For $AB \cdot BH = AH \cdot BH + BH^2$ (II. 2), and $AH^2 = AH \cdot AH$;
$\therefore\ AH \cdot AH = AH \cdot BH + BH^2$;
$\therefore\ AH \cdot AH$ is greater than $AH \cdot BH$.
Now these two rectangles have the same base AH; therefore the altitude AH must be greater than the altitude BH.

3. Let CH meet BF at K.
Triangles FAB, HAC are congruent; *I. 4*
$\therefore\ \angle\ FBA = \angle\ HCA$.
But $\angle\ HCA + \angle\ AHC =$ a right angle;
$\therefore\ \angle\ FBA + \angle\ AHC =$ a right angle;
$\therefore\ \angle\ FBA + \angle\ BHK =$ a right angle;
$\therefore\ \angle\ BKH =$ a right angle. *I. 32*

4. Let BE and CH intersect at O.
Because $EB = EF$, $\therefore\ \angle\ EBF = \angle\ EFB$. *I. 5*
But $\angle\ ABF$ is the complement of $\angle\ AFB$;
$\therefore\ \angle\ ABF$ is the complement of $\angle\ EBF$.
Now $\angle\ BOK$ is the complement of $\angle\ EBF$, by the preceding deduction;
$\therefore\ \angle\ BOK$, and consequently $\angle\ EOC$, $= \angle\ ABF$.
Again, because $\triangle$s ABF, ACH are congruent, *I. 4*
$\therefore\ \angle\ ABF = \angle\ ACH$;
$\therefore\ \angle\ EOC = \angle\ ECO$; $\therefore\ EO = EC$ (I. 6) $= EA$. *Const.*
Hence $\angle\ EAO + \angle\ ECO = \angle\ EOA + \angle\ EOC$ (*I.* 5) $= \angle\ AOC$;
$\therefore$ by the third deduction from I. 32, $\angle\ AOC$ is right.

5. Because $\angle\ F'AH'$ is right, and $\angle\ BAC$ is right;
$\therefore\ BA$ and AH' are in the same straight line. *I. 14*

6. Since $H'B \cdot BA = AH'^2$,

$\therefore$ $H'B$ is divided internally in medial section at A;

$\therefore$ AH' is greater than AB, by the second deduction from II. 11.

7. Let CH' meet BF' at K'.

Triangles $F'AB$, $H'AC$ are congruent; *I.* 4

$\therefore$ $\angle F'BA = \angle H'CA$.

But $\angle H'CA + \angle AH'C =$ a right angle;

$\therefore$ $\angle F'BA + \angle AH'C =$ a right angle;

$\therefore$ $\angle BK'H' =$ a right angle. *I.* 32

8. Let BE and CH' intersect at O'.

Because $EB = EF'$, $\therefore$ $\angle EBF' = \angle EF'B$. *I.* 5

But $\angle ABF'$ is the complement of $\angle AF'B$;

$\therefore$ $\angle ABF'$ is the complement of $\angle EBF'$.

Now $\angle BO'K'$ is the complement of $\angle EBF'$, by the preceding deduction;

$\therefore$ $\angle BO'K'$, and consequently $\angle EO'C$, $= \angle ABF'$.

Again, because $\triangle$s ABF', ACH' are congruent, *I.* 4

$\therefore$ $\angle ABF' = \angle ACH'$;

$\therefore$ $\angle EO'C = \angle ECO'$; $\therefore$ $EO' = EC$ (I. 6) $= EA$. *Const.*

Hence $\angle EAO' + \angle ECO' = \angle EO'A + \angle EO'C$ *I.* 5

$= \angle AO'C$;

$\therefore$ by the third deduction from I. 32, $\angle AO'C$ is right.

9. Because $HB \cdot BA = HA^2$,

$\therefore$ HB is divided externally in medial section at A;

and because $H'B \cdot BA = H'A^2$,

$\therefore$ $H'B$ is divided internally in medial section at A.

10. Since AB is divided internally in medial section at H, and AB, BH, HA are respectively $= CD$, DL, LC;

$\therefore$ CD is divided internally in medial section at L.

Since $CF \cdot FA = AB^2$ (II. 11) $= AC^2$,

$\therefore$ CF is divided internally in medial section at A; and consequently LG is divided internally in medial section at H.

Since CF, FA, AC are respectively $= AF'$, $F'C$, CA,

$\therefore$ AF' is divided at C, and consequently $H'G'$ at L', internally in medial section.

Since, by the preceding deduction, $H'B$ is divided internally in medial section at A,

$\therefore$ $L'D$ is divided internally in medial section at C.

The eight straight lines divided internally in medial section, are:

AB at H, CD at L, CF at A, LG at H, AF' at C, $H'G'$ at L', $H'B$ at A, and $L'D$ at C.

Hence the following eight straight lines are divided externally in medial section:

HB at A, LD at C, AF at C, HG at L, CF' at A, $L'G'$ at H', AB at H', and CD at L'.

PROPOSITION 12.

1. $BC^2 + CA^2 + 2\,BC \cdot CD = AB^2,$ *II.* 12
$\qquad = AC^2 + CB^2 + 2\,AC \cdot CE;$ *II.* 12
$\therefore\ 2\,BC \cdot CD = 2\,AC \cdot CE$, and $BC \cdot CD = AC \cdot CE$.

2. From D draw $DE \perp BC$ produced. *I.* 12
Then $\angle\ DCE$ = an angle of an equilateral triangle; *I.* 29
and $CD = 2\,CE$, by the eighth deduction from I. 9.
Hence $BD^2 = BC^2 + CD^2 + 2\,BC \cdot CE,$ *II.* 12
$\qquad = BC^2 + CD^2 + BC \cdot CD.$

3. $AC^2 + 3\,CD^2 = AB^2,$ *Hyp.*
$\qquad = BC^2 + CA^2 + 2\,BC \cdot CD;$ *II.* 12
$\therefore\quad 3\,CD^2 = BC^2 + 2\,BC \cdot CD;$
$\therefore\quad 4\,CD^2 = BC^2 + CD^2 + 2\,BC \cdot CD,$
$\qquad = BD^2;$ *II.* 4
$\therefore\quad 2\,CD = BD.$

4. The proposition becomes II. 4.

PROPOSITION 13.

1. $BC^2 + CA^2 - 2\,BC \cdot CD = AB^2,$ *II.* 13
$\qquad = AC^2 + CB^2 - 2\,AC \cdot CE;$ *II.* 13
$\therefore\ 2\,BC \cdot CD = 2\,AC \cdot CE$, and $BC \cdot CD = AC \cdot CE$.

2. From D draw $DE \perp BC$.
Then $\angle\ DCE$ = an angle of an equilateral triangle; *I.* 29
and $CD = 2\,CE$, by the eighth deduction from I. 9.
Hence $BD^2 = BC^2 + CD^2 - 2\,BC \cdot CE,$ *II.* 13
$\qquad = BC^2 + CD^2 - BC \cdot CD.$

3. $AC^2 + 3\,CD^2 = AB^2,$ *Hyp.*
$\qquad = BC^2 + CA^2 - 2\,BC \cdot CD;$ *II.* 13

$\therefore \quad 3\,CD^2 = BC^2 - 2\,BC\cdot CD;$

$\therefore \quad 4\,CD^2 = BC^2 + CD^2 - 2\,BC\cdot CD,$

$= BD^2;$ *II.* 7

$\therefore \quad 2\,CD = BD.$

4. The proposition becomes II. 7.

5. Let $\triangle\ ABC$ have AB^2 greater than $BC^2 + CA^2$:

it is required to prove $\angle\ C$ obtuse.

If $\angle\ C$ be right, $AB^2 = BC^2 + CA^2$; *I.* 47

and if $\angle\ C$ be acute, AB^2 is less than $BC^2 + CA^2$; *II.* 13

$\therefore$ $\angle\ C$ must be obtuse.

6. Let $\triangle\ ABC$ have AB^2 less than $BC^2 + CA^2$:

it is required to prove $\angle\ C$ acute.

If $\angle\ C$ be right, $AB^2 = BC^2 + CA^2$; *I.* 47

and if $\angle\ C$ be obtuse, AB^2 is greater than $BC^2 + CA^2$; *II.* 12

$\therefore$ $\angle\ C$ must be acute.

7. Let $\triangle\ ABC$ have $AB = AC$, and let BD be the projection of BC on AB:

it is required to prove $BC^2 = 2\,AB\cdot BD$.

$AC^2 = AB^2 + BC^2 - 2\,AB\cdot BD.$ *II.* 13

$\therefore BC^2 - 2\,AB\cdot BD = 0$, since $AB^2 = AC^2$;

$\therefore BC^2 = 2\,AB\cdot BD.$

PROPOSITION 14.

1. Let BHF be a semicircle (fig. to II. 14), HE a perpendicular to BF from any point H in the arc.

Bisect BF in G, and join GH.

Then $BE\cdot EF = GF^2 - GE^2$, *II.* 5

$= GH^2 - GE^2$,

$= HE^2$. *I.* 47, *Cor.*

2. Let AB be the given straight line, M a side of the given square. Bisect AB at C; find a square which is less than CB^2 by M^2, by the third deduction from I. 47; and from CB cut off CD equal to a side of this square.

AD, DB are the required segments.

For $AD\cdot DB = CB^2 - CD^2$, *II.* 5

$= M^2$.

M may be as small as we please, but it must not be greater than half of AB.

3. Let AB be the given straight line, M a side of the given square. Bisect AB at C; find a square which is greater than CB^2 by M^2, by the first deduction from I. 47; and from CB produced cut off CD equal to a side of this square.

AD, DB are the required segments.

For $AD \cdot DB = CD^2 - CB^2$, *II.* 6

$= M^2$.

M may be as small or as large as we please.

4. Let AD be the given straight line, M a side of the given square. From D draw $DE \perp AD$, and $= M$; join AE, and at E draw $EB \perp AE$, and meeting AD produced at B.

$AD \cdot DB$ is the required rectangle.

Bisect AB in C, and join CE.

Then $AD \cdot DB = CB^2 - CD^2$, *II.* 5

$= CE^2 - CD^2$, by the seventh deduction from I. 32,

$= DE^2$, *I.* 47, *Cor.*

$= M^2$.

DEDUCTIONS.

1. Let ABC be an isosceles triangle, having $AB = AC$, and from A let AD be drawn to cut BC internally or externally at D: to prove $BD \cdot DC = AB^2 - AD^2$ or $= AD^2 - AB^2$.

Draw $AE \perp BC$.

(1) $AB^2 - AD^2$

$= BE^2 - ED^2$, by the twelfth deduction from I. 47,

$= (BE + ED) \cdot (BE - ED)$, *II.* 5, *Cor.*

$= (BE + ED) \cdot (CE - ED)$,

$= BD \cdot DC$.

(2) $AD^2 - AB^2$

$= ED^2 - BE^2$, by the twelfth deduction from I. 47,

$= (ED + BE) \cdot (ED - BE)$, *II.* 5, *Cor.*

$= (ED + BE) \cdot (ED - EC)$,

$= BD \cdot DC$.

2. Let $ABCD$ be a $\|^m$, AC, BD its diagonals intersecting at E: to prove $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$.

Since the diagonals of a $\|^m$ bisect one another;

$\therefore\ AB^2 + BC^2 = 2\,AE^2 + 2\,BE^2$, *App. II.* 1

and $CD^2 + DA^2 = 2\,AE^2 + 2\,DE^2$. *App. II.* 1

Hence $AB^2 + BC^2 + CD^2 + DA^2$

$$= 4\,AE^2 + 2\,BE^2 + 2\,DE^2,$$
$$= 4\,AE^2 + 4\,BE^2,$$
$$= AC^2 + BD^2.$$

3. Let $ABCD$ be any quadrilateral, E, F, G, H the middle points of the sides AB, BC, CD, DA:
   it is required to prove $AC^2 + BD^2 = 2\,EG^2 + 2\,FH^2$.

   Join EF, FG, GH, HE.

   Then $EFGH$ is a $\|^m$, by the third deduction from I. 39;

   $\therefore\ EG^2 + FH^2 = EF^2 + FG^2 + GH^2 + EH^2$, by the second deduction.

   $$= 2\,EF^2 + 2\,EH^2;$$

   $\therefore\ 2\,EG^2 + 2\,FH^2 = 4\,EF^2 + 4\,EH^2,$

   $$= AC^2 + BD^2.$$

4. Let $ABCD$ be any quadrilateral, E and F the middle points of its diagonals AC, BD:
   to prove $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\,EF^2$.

   Join BE, ED.

   Then $AB^2 + BC^2 = 2\,AE^2 + 2\,EB^2$, *App. II.* 1

   and $CD^2 + DA^2 = 2\,AE^2 + 2\,ED^2$; *App. II.* 1

   $\therefore\ AB^2 + BC^2 + CD^2 + DA^2 = 4\,AE^2 + 2\,EB^2 + 2\,ED^2,$

   $= 4\,AE^2 + 4\,BF^2 + 4\,FE^2,$ *App. II.* 1

   $= AC^2 + BD^2 + 4\,FE^2.$

5. The sum of the squares on the sides of the triangle = twice the square on half the base together with twice the square on the radius of the circle; *App. II.* 1
   and the base of the triangle is constant, and so is the radius.

6. Let $ABCD$ be the $\|^m$, whose diagonals AC, BD intersect at E, and let F be any point on the $\bigcirc^{ce}$.

   Then $AF^2 + CF^2 = 2\,AE^2 + 2\,EF^2$, *App. II.* 1

   and $BF^2 + DF^2 = 2\,BE^2 + 2\,EF^2$; *App. II.* 1

   $\therefore\ AF^2 + BF^2 + CF^2 + DF^2 = 2\,AE^2 + 2\,BE^2 + 4\,EF^2,$

   = a constant.

7. For the sum of the squares of these distances is double of the sum of the squares of the two radii. *App. II.* 1

8. Let $\triangle\ ABC$ be right-angled at C, and D the middle point of the hypotenuse AB.

Then $2\,AD^2 + 2\,DC^2 = AC^2 + CB^2$, *App. II.* 1

$= AB^2$, *I.* 47

$= 4\,AD^2$;

$\therefore 2\,DC^2 = 2\,AD^2$, and $DC = AD = DB$.

9. Let ABC be any triangle, AH, BK, CL its three medians intersecting at G. *App. I.* 4

Then $2\,AB^2 + 2\,BC^2 = 4\,AK^2 + 4\,BK^2$, *App. II.* 1

$2\,BC^2 + 2\,CA^2 = 4\,BL^2 + 4\,CL^2$, *App. II.* 1

$2\,CA^2 + 2\,AB^2 = 4\,CH^2 + 4\,AH^2$; *App. II.* 1

$\therefore\ 4\,(AB^2 + BC^2 + CA^2)$

$= 4\,(AH^2 + BK^2 + CL^2) + 4\,AK^2 + 4\,BL^2 + 4\,CH^2$,

$= 4\,(AH^2 + BK^2 + CL^2) + CA^2 + AB^2 + BC^2$;

$\therefore 3\,(AB^2 + BC^2 + CA^2) = 4\,(AH^2 + BK^2 + CL^2)$.

Again, since $2\,AH = 3\,AG$, $2\,BK = 3\,BG$, $2\,CL = 3\,CG$, *App. I.* 4

$\therefore 4\,(AH^2 + BK^2 + CL^2) = 9\,(AG^2 + BG^2 + CG^2)$;

$\therefore\ 3\,(AB^2 + BC^2 + CA^2) = 9\,(AG^2 + BG^2 + CG^2)$.

10. $2\,PR^2 + AB^2 + CD^2$

$= 2\,PR^2 + 2\,AP^2 + 2\,AP^2 + 2\,DR^2 + 2\,CR^2$,

$= AR^2 + BR^2 + 2\,AP^2 + DR^2 + CR^2 + 2\,DR^2$,

$= 2\,AS^2 + 2\,RS^2 + 2\,BQ^2 + 2\,QR^2 + 2\,AP^2 + 2\,DR^2$,

$= \frac{1}{2}\,AD^2 + \frac{1}{2}\,AC^2 + \frac{1}{2}\,BC^2 + \frac{1}{2}\,BD^2 + \frac{1}{2}\,AB^2 + \frac{1}{2}\,CD^2$.

$2\,QS^2 + BC^2 + DA^2$

$= 2\,QS^2 + 2\,BQ^2 + 2\,BQ^2 + 2\,AS^2 + 2\,DS^2$,

$= BS^2 + CS^2 + 2\,BQ^2 + AS^2 + DS^2 + 2\,AS^2$,

$= 2\,AP^2 + 2\,PS^2 + 2\,BQ^2 + 2\,CR^2 + 2\,RS^2 + 2\,AS^2$,

$= \frac{1}{2}\,AB^2 + \frac{1}{2}\,BD^2 + \frac{1}{2}\,BC^2 + \frac{1}{2}\,CD^2 + \frac{1}{2}\,AC^2 + \frac{1}{2}\,AD^2$.

11. Let $ABCDE$ be any pentagon, F, G, H, K, L the middle points of the respective diagonals AC, BD, CE, DA, EB: if FG, GH, HK, KL, LF be joined, it is required to prove $3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) = AC^2 + BD^2 + CE^2 + DA^2 + EB^2 + 4\,(FG^2 + GH^2 + HK^2 + KL^2 + LF^2)$.

From the quadrilateral $ABCD$,

$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\,FG^2$, by the fourth deduction;

from the quadrilateral $BCDE$,

$BC^2 + CD^2 + DE^2 + EB^2 = BD^2 + CE^2 + 4\,GH^2$;

from the quadrilateral $CDEA$,

$CD^2 + DE^2 + EA^2 + AC^2 = CE^2 + DA^2 + 4HK^2$;

from the quadrilateral $DEAB$,

$DE^2 + EA^2 + AB^2 + BD^2 = DA^2 + EB^2 + 4\,KL^2$;

from the quadrilateral $EABC$,
$EA^2 + AB^2 + BC^2 + CE^2 = EB^2 + AC^2 + 4\,LF^2$.
Add these results together; then $3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) + (AC^2 + BD^2 + CE^2 + DA^2 + EB^2)$,
$= 2\,(AC^2 + BD^2 + CE^2 + DA^2 + EB^2) + 4\,(FG^2 + GH^2 + HK^2 + KL^2 + LF^2)$;
$\therefore\ 3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) = AC^2 + BD^2 + CE^2 + DA^2 + EB^2 + 4\,(FG^2 + GH^2 + HK^2 + KL^2 + LF^2)$.

[See *Solutions of the Principal Questions of Dr Hutton's Course of Mathematics*, by Thomas Stephens Davies (1840), p. 313.]

12. Let C be the middle point of AB, and O any other point; join OA, OB, OC.

Then $OA^2 + OB^2 = 2\,AC^2 + 2\,OC^2$. *App. II.* 1

Now $2\,AC^2$ is a fixed magnitude;
$\therefore\ OA^2 + OB^2$ will increase or diminish as $2\,OC^2$ increases or diminishes;
$\therefore\ OA^2 + OB^2$ will be the least when $2\,OC^2$ is least, that is, when O coincides with C.

13. Because $AC = AE$, $\therefore\ \angle\,ACE$ = half of a right angle, and $EC^2 = 2\,AC^2$.
Because $BC = DF$, and $BD = FG$,
$\therefore\ BC \mp BD = DF \mp FG$, that is, $DC = DG$;
$\therefore\ \angle\,DCG$ = half of a right angle, and $CG^2 = 2\,CD^2$.
Because $\angle$s ACE, DCG are each = half of a right angle,
$\therefore\ \angle\,ECG$ is right.
Now $AD^2 + DB^2 = EF^2 + FG^2$,
$= EG^2$, *I.* 47
$= EC^2 + CG^2$, *I.* 47
$= 2\,AC^2 + 2\,CD^2$.

These figures may be derived from those in the text by rotating through a semi-revolution round AE as an axis the triangle CAE, and round EF as an axis the triangle GEF.

14. (1) In the fig. on p. 302 of *Euclid*, find E the middle point of BC, and join AE. The proof will be the same as in the fourth deduction from II. 14.
Or thus, without finding E:
$BC^2 = BA^2 + AC^2$, *I.* 47
$= (BD^2 + DA^2) + (CD^2 + DA^2)$, *I.* 47
$= BD^2 + CD^2 + 2\,DA^2$.

But $BC^2 = BD^2 + CD^2 + 2\,BD \cdot CD$; *II.* 4

$\therefore\ BD^2 + CD^2 + 2\,DA^2 = BD^2 + CD^2 + 2\,BD \cdot CD$;

$\therefore\ DA^2 = BD \cdot CD.$

(2) $BD^2 = BD \cdot BC - BD \cdot CD$, *II.* 3

$= BD \cdot BC - AD^2$; by (1)

$\therefore\ BD^2 + AD^2 = BD \cdot BC$;

$\therefore\ AB^2 = BD \cdot BC.$ *I.* 47

15. See the seventh deduction from II. 7.

16. Let A, B, C be three unequal straight lines :
it is required to prove
$A^2 + B^2 + C^2$ greater than $A \cdot B + B \cdot C + C \cdot A$.

$A^2 + B^2$ is greater than $2\,A \cdot B$,

$B^2 + C^2$ is greater than $2\,B \cdot C$,

and $C^2 + A^2$ is greater than $2\,C \cdot A$, by the fifteenth deduction;

$\therefore\ 2\,(A^2 + B^2 + C^2)$ is greater than $2\,(A \cdot B + B \cdot C + C \cdot A)$;

$\therefore\ A^2 + B^2 + C^2$ is greater than $A \cdot B + B \cdot C + C \cdot A$.

17. Let A, B, C be the three unequal straight lines :
it is required to prove
$(A + B + C)^2$ greater than $3\,(A \cdot B + B \cdot C + C \cdot A)$.

$(A + B + C)^2 = A^2 + B^2 + C^2 + 2\,A \cdot B + 2\,B \cdot C + 2\,C \cdot A$, by the tenth deduction from II. 4;

$\therefore\ (A + B + C)^2$ is greater than $A \cdot B + B \cdot C + C \cdot A + 2\,A \cdot B + 2\,B \cdot C + 2\,C \cdot A$; by the sixteenth deduction,

$\therefore\ (A + B + C)^2$ is greater than $3\,(A \cdot B + B \cdot C + C \cdot A)$.

18. Let ABC be a triangle : it is required to prove
$AB^2 + BC^2 + CA^2$ less than $2\,(AB \cdot BC + BC \cdot CA + CA \cdot AB)$.

AB is less than $BC + CA$; *I.* 20

$\therefore\ AB^2$ is less than $AB \cdot BC + AB \cdot CA$.

Similarly BC^2 is less than $BC \cdot AB + BC \cdot CA$,

and CA^2 is less than $CA \cdot AB + CA \cdot BC$;

$\therefore\ AB^2 + BC^2 + CA^2$ is less than $2\,(AB \cdot BC + BC \cdot CA + CA \cdot AB)$.

19. Let ABC be a triangle, and let BE, CF be the medians drawn from B and C: if AC is greater than AB, it is required to prove BE less than CF.

$AC^2 + BC^2 = 2\,AF^2 + 2\,CF^2$,

and $AB^2 + BC^2 = 2\,AE^2 + 2\,BE^2$. *App. II.* 1

But $AC^2 + BC^2$ is greater than $AB^2 + BC^2$;

$\therefore\ 2AF^2 + 2\,CF^2$ is greater than $2\,AE^2 + 2\,BE^2$.

Now $2\,AF^2$ is less than $2\,AE^2$, since AF is less than AE;
$\therefore$ $2\,CF^2$ is greater than $2\,BE^2$;
$\therefore$ BE is less than CF.

[Another proof of this theorem is given in Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire* (6ème éd.), pp. 2, 3.]

20. $AD \cdot DB - AE \cdot EB = (CB^2 - CD^2) - (CB^2 - CE^2)$, *II.* 5
$= CE^2 - CD^2$,
$= (CE + CD) \cdot (CE - CD)$, *II.* 5, *Cor.*
$= (CE + CD) \cdot DE$,
$= CE \cdot DE + CD \cdot DE$;
$\therefore$ $AD \cdot DB = AE \cdot EB + CE \cdot DE + CD \cdot DE$.

[It should perhaps have been stated that D and E lie on the same side of C.]

21. Bisect $\angle ABC$ by BE.
Then $BE = AE$, $BE = BC$, and $ED = CD$.
Because $AC^2 = AB^2 = AD^2 + DB^2$. *I.* 47
and $AE^2 = BC^2 = DC^2 + DB^2$; *I.* 47
$\therefore$ $AC^2 + AE^2 = AD^2 + DC^2 + 2\,DB^2$.
But $AC^2 + AE^2 = 2\,AD^2 + 2\,DC^2$; *II.* 10
$\therefore$ $2\,AD^2 + 2\,DC^2 = AD^2 + DC^2 + 2\,DB^2$;
$\therefore$ $AD^2 + DC^2 = 2\,DB^2$.

22. Let CD be the given straight line.
Find the side of a square $= 3\,CD^2$, by the second deduction from I. 47; from DC produced cut off $DA =$ this side, and from CD cut off $CB = CA$.
CD is divided internally at B as was required.
For $AD^2 + DB^2 = 2\,CB^2 + 2\,CD^2$; *II.* 10
$\therefore$ $3\,CD^2 + DB^2 = 2\,CB^2 + 2\,CD^2$;
$\therefore$ $CD^2 + DB^2 = 2\,CB^2$.

23. (1) $AH^2 = AB \cdot BH = (BH + AH) \cdot BH$,
$= BH^2 + AH \cdot BH$;
$\therefore$ $AH \cdot BH = AH^2 - BH^2$,
$= (AH + BH) \cdot (AH - BH)$. *II.* 5, *Cor.*
$AH'^2 = AB \cdot BH'$,
$= (BH' - AH') \cdot BH'$,
$= BH'^2 - AH' \cdot BH'$;
$\therefore$ $AH' \cdot BH' = BH'^2 - AH'^2$,
$= (BH' + AH') \cdot (BH' - AH')$. *II.* 5, *Cor.*

$$(2)\ AH\cdot(AH-BH)=AH^2-AH\cdot BH,$$
$$=AH^2-(AH^2-BH^2),\qquad \text{by (1)}$$
$$=BH^2.$$

$$AH'\cdot(AH'+BH')=AH'^2+AH'\cdot BH',$$
$$=AB\cdot BH'+AH'\cdot BH',$$
$$=BH'\cdot(AB+AH'),$$
$$=BH'^2.$$

$$(3)\ AB^2+BH^2=(BH+AH)^2+BH^2,$$
$$=2\,BH^2+2\,BH\cdot AH+AH^2,\qquad II.\ 4$$
$$=2\,BH\cdot(BH+AH)+AH^2,$$
$$=2\,BH\cdot AB+AH^2,$$
$$=2\,AH^2+AH^2,$$
$$=3\,AH^2.$$

$$AB^2+BH'^2=(BH'-AH')^2+BH'^2,$$
$$=2\,BH'^2-2\,BH'\cdot AH'+AH'^2,\qquad II.\ 7$$
$$=2\,BH'\cdot(BH'-AH')+AH'^2,$$
$$=2\,BH'\cdot AB+AH'^2,$$
$$=2\,AH'^2+AH'^2,$$
$$=3\,AH'^2.$$

$$(4)\ (AB+BH)^2=AB^2+BH^2+2\,AB\cdot BH,\qquad II.\ 4$$
$$=3\,AH^2+2\,AH^2,\qquad \text{by (3)}$$
$$=5\,AH^2.$$

$$(AB+BH')^2=AB^2+BH'^2+2\,AB\cdot BH',\qquad II.\ 4$$
$$=3\,AH'^2+2\,AH'^2,\qquad \text{by (3)}$$
$$=5\,AH'^2.$$

$$(5)\ (AH-BH)^2=AH^2+BH^2-2\,AH\cdot BH,\qquad II.\ 7$$
$$=AH^2+BH^2-2\,(AH^2-BH^2),\qquad \text{by (1)}$$
$$=AH^2+BH^2-2\,AH^2+2\,BH^2,$$
$$=3\,BH^2-AH^2.$$

$$(BH'-AH')^2=BH'^2+AH'^2-2\,BH'\cdot AH',\qquad II.\ 7$$
$$=BH'^2+AH'^2-2\,(BH'^2-AH'^2),\ \text{by (1)}$$
$$=BH'^2+AH'^2-2\,BH'^2+2\,AH'^2,$$
$$=3\,AH'^2-BH'^2.$$

$$(6)\ (AH+BH)^2=AH^2+BH^2+2\,AH\cdot BH,\qquad II.\ 4$$
$$=AH^2+BH^2+2\,(AH^2-BH^2),\ \text{by (1)}$$
$$=AH^2+BH^2+2\,AH^2-2\,BH^2,$$
$$=3\,AH^2-BH^2.$$

$$
\begin{aligned}
&(AH' + BH')^2 \\
&\quad = AH'^2 + BH'^2 + 2\,AH' \cdot BH', && \textit{II. } 4 \\
&\quad = AH'^2 + BH'^2 + 2\,(BH'^2 - AH'^2), && \text{by (1)} \\
&\quad = AH'^2 + BH'^2 + 2\,BH'^2 - 2\,AH'^2, \\
&\quad = 3\,BH'^2 - AH'^2.
\end{aligned}
$$

(7)
$$
\begin{aligned}
&(AB + AH)^2 \\
&\quad = AB^2 + AH^2 + 2\,AB \cdot AH, && \textit{II. } 4 \\
&\quad = (AH + BH)^2 + AH^2 + 2\,(AH + BH) \cdot AH, \\
&\quad = 4\,AH^2 + BH^2 + 4\,AH \cdot BH, && \textit{II. } 4 \\
&\quad = 4\,AH^2 + BH^2 + 4\,(AH^2 - BH^2), && \text{by (1)} \\
&\quad = 8\,AH^2 - 3\,BH^2.
\end{aligned}
$$

$$
\begin{aligned}
&(AH' - AB)^2 \\
&\quad = AB^2 + AH'^2 - 2\,AB \cdot AH', && \textit{II. } 7 \\
&\quad = (BH' - AH')^2 + AH'^2 - 2\,(BH' - AH') \cdot AH', \\
&\quad = 4\,AH'^2 + BH'^2 - 4\,AH' \cdot BH', && \textit{II. } 7 \\
&\quad = 4\,AH'^2 + BH'^2 - 4\,(BH'^2 - AH'^2), && \text{by (1)} \\
&\quad = 8\,AH'^2 - 3\,BH'^2.
\end{aligned}
$$

(8)
$$
\begin{aligned}
AB^2 + AH^2 &= (AH + BH)^2 + AH^2, \\
&= 3\,AH^2 - BH^2 + AH^2, && \text{by (6)} \\
&= 4\,AH^2 - BH^2.
\end{aligned}
$$

$$
\begin{aligned}
AB^2 + AH'^2 &= (BH' - AH')^2 + AH'^2, \\
&= 3\,AH'^2 - BH'^2 + AH'^2, && \text{by (5)} \\
&= 4\,AH'^2 - BH'^2.
\end{aligned}
$$

24.
$$
\begin{aligned}
BC^2 &= AB^2 + AC^2 \pm 2\,AB \cdot AQ, && \textit{II. } 12, 13 \\
BC^2 &= AB^2 + AC^2 \pm 2\,AC \cdot AP; && \textit{II. } 12, 13 \\
\therefore\ 2\,BC^2 &= 2\,AB^2 \pm 2\,AB \cdot AQ + 2\,AC^2 \pm 2\,AC \cdot AP, \\
&= 2\,AB \cdot (AB \pm AQ) + 2\,AC \cdot (AC \pm AP), \\
&= 2\,AB \cdot BQ + 2\,AC \cdot CP.
\end{aligned}
$$

25. Let $\triangle\ ABC$ be right-angled at C, and from the hypotenuse AB let there be cut off $AD = AC$, and $BE = BC$: to prove $DE^2 = 2\,AE \cdot DB$.

By the tenth deduction from II. 4 it is proved that

$$
AB^2 = AE^2 + ED^2 + DB^2 + 2\,AE \cdot ED + 2\,AE \cdot DB + 2\,ED \cdot DB
$$

$$
\begin{aligned}
&\therefore\ AB^2 + DE^2, \\
&= (AE^2 + ED^2 + 2\,AE \cdot ED) + (DB^2 + DE^2 + 2\,ED \cdot DB) + 2\,AE \cdot DB \\
&= AD^2 + BE^2 + 2\,AE \cdot DB, && \textit{II. } 4
\end{aligned}
$$

$= AC^2 + BC^2 + 2\,AE \cdot DB,$

$= \qquad AB^2 + 2\,AE \cdot DB;$ *I.* 47

$\therefore DE^2 = 2\,AE \cdot DB.$

Since $\frac{1}{2}\,DE^2 = AE \cdot DB$, express DE by an even whole number, and resolve half of its square into two factors AE, DB;

then $AE + DE$, $DB + DE$, and $AE + DE + DB$ will represent the two sides and the hypotenuse.

Let $DE = 2$; then $AE \cdot DB = 2 = 1 \times 2$.

If $AE = 1$, $DB = 2$, $DE = 2$,

$AE + DE = 3$, $DB + DE = 4$, $AE + DE + DB = 5$.

Let $DE = 4$; then $AE \cdot DB = 8 = 2 \times 4$ or 1×8.

If $AE = 2$, $DB = 4$, $DE = 4$,

$AE + DE = 6$, $DB + DE = 8$, $AE + DE + DB = 10$;

if $AE = 1$, $DB = 8$, $DE = 4$,

$AE + DE = 5$, $DB + DE = 12$, $AE + DE + DB = 13$;

and so on.

LOCI.

1. Since $\triangle\ ABC$ is given, its sides AB, AC are given;

$\therefore$ the sum of the squares of AB and AC is given.

Find M such that $M^2 = AB^2 + AC^2$; *I.* 47

the question is now reduced to App. II. 4.

2. Since $\triangle\ ABC$ is given, its sides AB, AC are given;

$\therefore$ the difference of the squares of AB and AC is given.

Find M such that $M^2 = AB^2 - AC^2$, or $AC^2 - AB^2$, by the third deduction from I. 47;

the question is now reduced to App. II. 5.

3. Of all the triangles whose base is the given base BC, and the sum of whose sides is given, let ABC be one; and let CD drawn $\perp$ AC meet the bisector of the exterior vertical angle at D.

From D draw $DE \perp BA$ produced, and join BD.

Then $\triangle$s ACD, AED are congruent; *I.* 26

$\therefore AC = AE$, and $DC = DE$;

$\therefore BE = BA + AC$, and is consequently given.

Now $BE^2 = BD^2 - DE^2,$ *I.* 47, *Cor.*

$= BD^2 - DC^2;$

$\therefore BD^2 - DC^2$ is given;

$\therefore$ the locus of D is a straight line. *App. II.* 5

4. Of all the triangles whose base is the given base BC, and the difference of whose sides is given, let ABC be one; and let CD drawn $\perp$ AC meet the bisector of the interior vertical angle at D.

From D draw $DE \perp AB$ or AB produced, and join BD.

Then $\triangle$s ACD, AED are congruent; *I.* 26

$\therefore$ $AC = AE$, and $DC = DE$;

$\therefore$ $BE = AB - AC$ or $AC - AB$, and is consequently given.

Now $BE^2 = BD^2 - DE^2$, *I.* 47, *Cor.*

$= BD^2 - DC^2$;

$\therefore$ $BD^2 - DC^2$ is given;

$\therefore$ the locus of D is a straight line. *App. II.* 5

5. Let AB be the variable chord, E the fixed point situated either inside or outside the given circle. Let F be the middle point of AB, and O the centre of the given circle; join OA, OE, OF, EF.

Since $\angle$ AEB is right;

$\therefore$ $FE = FA$, by the seventh deduction from I. 32;

and since $AF = BF$,

$\therefore$ OF is $\perp$ AB, by the first deduction from I. 8.

Hence $OF^2 + FE^2 = OF^2 + FA^2$,

$= OA^2$, *I.* 47

$=$ a constant magnitude.

Now OFE is a triangle whose base OE is fixed, and the sum of the squares of whose sides OF, FE is constant;

$\therefore$ the locus of F is the $\bigcirc^{ce}$ of a circle whose centre is the middle point of OE. *App. II.* 4

When the fixed point E is on the $\bigcirc^{ce}$ of the given circle, F the middle point of AB, being always equidistant from A, E, B, by the seventh deduction from I. 32, may be proved to be the centre of the given circle;

Hence in this case the locus reduces to a point.

When the fixed point E is inside the circle, the middle point of AB describes the entire circle whose centre is the middle point of OE; when E is outside the circle, the middle point of AB describes only part of the circle whose centre is the middle point of OE; and when E is on the $\bigcirc^{ce}$, the middle point of AB remains fixed. To obtain a clearer idea

of the connection of the three cases, suppose AE, BE to meet the $\bigcirc^{ce}$ of the given circle again at C and D.

In the third case C and D will be coincident with E. Then the middle points of AB, BC, CD, DA will describe the entire circle whose centre is the middle point of OE.

BOOK III.

PROPOSITION 1.

1. In the given circle draw two chords not parallel. The straight lines which bisect the chords perpendicularly will intersect at the centre of the circle.

2. Take any point in the arc, and join it with the ends of the arc. The straight lines bisecting these two chords perpendicularly will intersect at the required centre.

3. Let A, B, C be the three points.
   Join AB, AC, and bisect them perpendicularly by DF, EF intersecting at F. F is the centre of the required circle.
   The problem is impossible when DF, EF do not intersect, but are parallel; this will be the case when the three points A, B, C are in the same straight line.

4. Let A and B be the given points; CD the given straight line.
   Join AB, and bisect it perpendicularly by EF which meets CD at F. F is the centre of the required circle.
   For $FA = FB$. *I.* 4
   The problem is impossible when EF, CD do not intersect but are parallel; this will be the case when CD is $\perp$ AB or AB produced.

5. Let A and B be the given points, M the given straight line.
   Join AB, and bisect it perpendicularly by CD. With A or B as centre and M as radius cut CD at the points E, F.
   E or F is the centre of a circle such as is required.
   The problem is impossible when the circle with A or B as centre and M as radius will not cut CD; this will be the case when M is less than half AB.

6. They are concurrent at the centre of the circle. *III.* 1, *Cor.* 1

7. Let A be the point outside the circle BCD, and let AB, AC drawn to the $\bigcirc^{ce}$ be equal.

   Join BC.

   Then $\triangle$ ABC is isosceles, and the bisector of $\angle$ BAC will bisect BC, a chord of the circle, perpendicularly;

   $\therefore$ this bisector will pass through the centre of the circle.

8. The reasoning is the same as in the preceding deduction.

9. Take any point outside or inside a circle, or on its $\bigcirc^{ce}$, and from it draw two equal straight lines to the $\bigcirc^{ce}$. Bisect the angle contained by these two straight lines, and a diameter will be obtained. This diameter may be bisected, or another point may be chosen, and another diameter obtained. The two diameters intersect at the centre.

PROPOSITION 2.

1. Let ABC be a circle (fig. to III. 2), D its centre, and, if possible let the straight line AB cut the $\bigcirc^{ce}$ at A, E, B.

   Draw the three radii DA, DE, DB.

   Since $DA = DB$, $\therefore \angle DAB = \angle DBA$; *I.* 5

   Since $DA = DE$, $\therefore \angle DAE = \angle DEA$; *I.* 5

   $\therefore \angle DEA = \angle DBA$, which is impossible. *I.* 16

   Hence AB cannot cut the $\bigcirc^{ce}$ at more than two points.

2. Let A be the given point, BC the first given straight line, D the second.

   With A as centre and D as radius describe a circle cutting BC or BC produced at the points E, F.

   E or F is the centre of a circle such as is required. Two circles may be drawn when the circle whose centre is A and radius D meets BC at two points; when it meets BC at only one point, only one circle can be drawn; and when it does not meet BC at all, no circle can be drawn. These three cases will happen when D is greater than, equal to, or less than the perpendicular drawn from A to BC.

PROPOSITION 3.

1. No. Both may be inside the circle, or both outside, or one inside and one outside.

2. The sides of the triangle are chords of the circle;
   $\therefore$ the perpendiculars to them from the centre will bisect them.

3. Let $ACDB$ be a straight line cutting two concentric circles, the outer one in A and B, and the inner one in C and D:
   to prove $AC = BD$.
   Find O the common centre, and draw $OE \perp AB$.
   Then $AE = BE$, and $CE = DE$; *III.* 3
   $\therefore AE - CE = BE - DE$; $\therefore AC = BD$.

4. Let P be the point inside the circle.
   Find O the centre of the circle, join OP, and draw $APB \perp OP$ and meeting the $\bigcirc^{ce}$ in A and B.
   Then OP bisects AB. *III.* 3

5. Let AB and CD be two parallel chords in a circle.
   The diameter which bisects AB is also $\perp AB$;
   $\therefore$ it is also $\perp CD$, by the second deduction from I. 29;
   $\therefore$ it also bisects CD.

6. Draw any two parallel chords in the circle; join their middle points, and a diameter will be obtained. This diameter may be bisected, or another pair of parallel chords may be drawn, and another diameter obtained.

7. Let OAB be an isosceles triangle, and with centre O let a circle be described cutting the base AB in C and D:
   to prove $AC = BD$.
   Draw $OE \perp AB$.
   Then $AE = BE$, by the third deduction from I. 26, and $CE = DE$; *III.* 3
   $\therefore AE - CE = BE - DE$; $\therefore AC = BD$.

8. Let two circles cut each other at A and B, and through A and B let there be drawn the two parallels CAD, EBF meeting the circles at C, D and E, F respectively.
   Find G and H the centres of the two circles, and draw KGM, $LHN \perp CD$ and EF, and therefore $\parallel$ each other.

Then $KMNL$ is a $\|^m$; $\therefore$ $KL = MN$.
Now $KL = KA + LA = \frac{1}{2} AC + \frac{1}{2} AD = \frac{1}{2} CD$;
and $MN = MB + NB = \frac{1}{2} EB + \frac{1}{2} BF = \frac{1}{2} EF$;
$\therefore$ $CD = EF$.

9. In the figure to the preceding deduction let CAD, $E'AF'$ be the two straight lines drawn through A, and making equal angles with GH the line of centres.

Since GH bisects AB perpendicularly, by the tenth deduction from I. 8; the part of the figure on the same side of GH as A would, if it were rotated round GH as an axis, coincide with the part on the same side as B; and $E'AF'$ would occupy the position of EBF, because they make equal angles with GH. Hence this deduction is reduced to the preceding.

PROPOSITION 4.

1. They must both be diameters.
2. Let $ABCD$ be a $\|^m$ whose vertices A, B, C, D are on the $\bigcirc^{ce}$ of a circle.

Since its diagonals AC, BD bisect one another, by the tenth deduction from I. 29;
$\therefore$ AC and BD must be diameters of the circle, and consequently equal.

3. Every $\|^m$ which has its vertices on the $\bigcirc^{ce}$ of a circle, has its diagonals equal;
$\therefore$ it must be a rectangle, by the seventh deduction from I. 34.

PROPOSITION 5.

1. Yes. They may be concentric.
2. It may be a diameter of one of them, but not a diameter of both, because if it were a diameter of both, the middle point of this diameter would be the centre of both circles.
3. Let two circles intersect at A and B, and let AB be a diameter of one of them.
Then C, the middle point of AB, is the centre of this circle.
Let D be the centre of the other circle and join DC.

Since AB is a chord of the second circle, and DC drawn through the centre bisects it,

$\therefore$ DC is $\perp$ AB. *III.* 3

4. Let C be one of the points of intersection of two circles whose centres are A and B.

Join AC, BC.

Then the sum of AC and BC is greater than AB, *I.* 20
and the difference of AC and BC is less than AB. *I.* 20, *Cor.*

5. If the distance between the centres of two circles be less than the sum, and greater than the difference of their radii, the two circles will cut one another.

Let A and B be the centres of the two circles, a and b their radii. Denote AB by c, and suppose a to be greater than b.

With centre A and radius a describe a circle cutting AB or AB produced at D; with centre B and radius b describe a circle cutting BA or BA produced at E.

Then BE will partially overlap AD, and consequently one point E of the circle whose centre is B will fall inside the circle whose centre is A.

Let the circle whose centre is B cut AB produced at E'.

Since c is greater than $a - b$; $\therefore$ $c + b$ is greater than a;

$\therefore$ AE' is longer than AD, and consequently the point E' will fall outside the circle whose centre is A.

Hence, since circles are continuous curves, the circle whose centre is B will cut the circle whose centre is A.

PROPOSITION 6.

1. Since the centre of a circle is inside the circle, and since two circles which touch externally are each outside the other, two such circles cannot have the same centre.

2. If two circles meet each other at any point, they cannot have the same centre.

3. Let the circle whose centre is B fall inside the circle whose centre is A.

Join AB, and produce it to meet the two circles at C and D.

Then AD and BC are the two radii,

and $AD - BC = AB + CD$;

$\therefore$ AB is less than $AD - BC$ by CD.

4. Let the centres of the two circles be A and B.

Join AB, and let it meet the two circles at D and C.

Then AD and BC are the two radii,

and $AD + BC = AB - CD$;

$\therefore$ AB is greater than $AD + BC$ by CD.

5. If the distance between the centres of two circles be less than the difference of their radii, one of the circles will be inside the other, and not touch it.

Let A and B be the centres of the two circles, a and b their radii. Denote AB by c, and suppose a to be greater than b.

With centre A and radius a describe a circle cutting AB produced at D; with centre B and radius b describe a circle cutting AB produced at E.

Since $a - b$ is greater than c; $\therefore$ a is greater than $c + b$;

$\therefore$ AD is longer than AE, and consequently the point E will fall inside the circle whose centre is A.

If F be any other point on the circle whose centre is B, then AF is less than $AB + BF$; *I.* 20

$\therefore$ AF is less than AE, and consequently than AD.

Hence every point on the circle whose centre is B must be inside the circle whose centre is A. *III. Def.* 1, *Cor.* 2

If the distance between the centres of two circles be greater than the sum of their radii, the circles will be outside each other and will not touch.

The proof of this may be left to the reader.

PROPOSITION 7.

1. Because $OP + PC$ is greater than OC, and $OC = OD$;

$\therefore$ $OP + PC$ is greater than OD, or $OP + PD$;

$\therefore$ PC is greater than PD.

2. Let P be any point inside the circle ABC (fig. to III. 7);

then PA is the greatest straight line that can be drawn from P to the $\bigcirc^{ce}$, and PD is the least.

Now $PA + PD =$ a diameter of the circle, which is a constant length.

3. If from AD there be cut off $AE = DP$, then E will be a point such as is required.

For $EA = PD$, and $\therefore$ $ED = PA$.

It is easy to show that $OE = OP$; and therefore if on any diameter there be taken points F, G, &c., such that OF, OG, &c. $= OP$, the points F, G, &c. will also be points satisfying the requirement.
Since $OE = OF = OG =$ &c. $= OP$, the points E, F, G, &c. will lie on the $\bigcirc^{ce}$ of a circle whose centre is O and radius OP.

4. In $\triangle$s POA, POB, $\begin{cases} PO = PO \\ OA = OB \\ \angle\ POA \text{ is greater than } \angle\ POB; \end{cases}$

$\therefore PA$ is greater than PB. *I.* 24

In $\triangle$s POC, POD, $\begin{cases} PO = PO \\ OC = OD \\ \angle\ POC \text{ is greater than } \angle\ POD; \end{cases}$

$\therefore PC$ is greater than PD. *I.* 24

PROPOSITION 8.

1. $PE + OE$ is greater than OP, that is, than $PD + OD$; *I.* 20
   and $OE = OD$; $\therefore PE$ is greater than PD.
2. $PF + OF$ is greater than $PE + OE$; *I.* 21
   and $OF = OE$; $\therefore PF$ is greater than PE.
3. Let P be any point outside the circle ABC (fig. to III. 8); then PA is the greatest straight line that can be drawn from P to the $\bigcirc^{ce}$, and PD is the least.
   Now $PA - PD =$ a diameter of the circle, which is a constant length.
4. If the point P, which may be situated either inside or outside the circle, gradually approach the $\bigcirc^{ce}$, the straight line PA gradually approaches in magnitude to the diameter of the circle, and PD gradually diminishes. If P at last reaches the $\bigcirc^{ce}$, PA becomes a diameter, and PD vanishes.
   Hence, when P is on the $\bigcirc^{ce}$, either the sum or the difference of the greatest and least straight lines that can be drawn from it to the $\bigcirc^{ce}$ is constant.
5. $AD = OA + OD = OB + OE$; and $OB + OE$ is greater than BE. *I.* 20

   Again, in $\triangle$s BOE, COF, $\begin{cases} BO = CO \\ OE = OF \\ \angle\ BOE \text{ is greater than } \angle\ COF; \end{cases}$

   $\therefore BE$ is greater than CF. *I.* 24

6. PFC would at length become a tangent to the circle.

7. Suppose the tangent from P to the circle to be PL.
   Then since PL may be considered as drawn from P to the concave part of the ⊙^{ce}, PL is less than PC; and since PL may also be considered as drawn from P to the convex part of the ⊙^{ce}, PL is greater than PF.

8. If PFC revolved round P, clockwise till F and C coincided at L, and anti-clockwise till F and C coincided at L', then PL and PL' would be two tangents drawn to the circle from P, and the straight line LL' would separate the concave from the convex part of the ⊙^{ce} viewed from P.

PROPOSITION 9.

Since from D, a point inside the circle, two equal straight lines DA, DB are drawn to the ⊙^{ce}, the bisector of $\angle ADB$ passes through the centre. Let DE be that bisector.
Similarly, if DF be the bisector of $\angle BDC$, DF passes through the centre.
Hence D must be the centre.

PROPOSITION 10.

1. Repeat the proposition, substituting for 'cut,' the word 'meet,' and for the authority III. 5, the second deduction from III. 6.

2. They must coincide.

3. Suppose the circle ABC to remain the same, but that the radius of circle EBC increases in length. If C and D continue to be the points in which the circle EBC cuts the circle ABC, the arc CD will grow less and less curved as the radius of EBC increases, and finally, if the radius of EBC be regarded as infinitely long, the arc CD may be regarded as a straight line.
   Hence, since the circle EBC cannot cut the circle ABC in more than two points, the straight line CD cannot do so either.

PROPOSITION 11.

1. Let A and B be the centres of two circles which touch internally at C.

   Join AB and produce it to C. *III.* 11

   Then AC and BC are radii of the two circles,

   and $AC - BC = AB$.

2. Let A and B be the centres of two circles which touch internally at C. Through C let CED be drawn cutting the two $\bigcirc^{ces}$ at E and D. AD shall be $\parallel BE$.

   Join AB and produce it to C. *III.* 11

   Since $BC = BE$, $\therefore \angle BCE = \angle BEC$; *I.* 5

   since $AC = AD$, $\therefore \angle ACD = \angle ADC$. *I.* 5

   But $\angle BCE = \angle ACD$; $\therefore \angle BEC = \angle ADC$;

   $\therefore AD$ is $\parallel BE$. *I.* 28

3. Let two circles whose centres are F and G cut each other at the points B and C. Suppose the two circles to remain constant in size, but to move so that their centres may approach each other; then the points of intersection B and C will approach each other, and at length coincide. But since the straight line FG (or FG produced) always bisects BC, FG must pass through the point in which B and C coincide, that is, through the point of contact of the two circles.

PROPOSITION 12.

1. Let A and B be the centres of two circles which touch externally at C.

   Join AB, which passes through C. *III.* 12

   Then AC and BC are radii of the two circles,

   and $AC + BC = AB$.

2. Let A and B be the centres of two circles which touch externally at C. Through C let ECD be drawn cutting the two $\bigcirc^{ces}$ at E and D. AD shall be $\parallel BE$.

   Since $BC = BE$, $\therefore \angle BCE = \angle BEC$; *I.* 5

   since $AC = AD$, $\therefore \angle ACD = \angle ADC$. *I.* 5

   But $\angle BCE = \angle ACD$; $\therefore \angle BEC = \angle ADC$;

   $\therefore AD$ is $\parallel BE$. *I.* 27

3. Repeat the proof of the third deduction from III. 11, but suppose the centres of the two circles to recede from each other.

PROPOSITION 13.

1. Let A and B be the centres of two circles, and let AB be = the sum of the two radii.

   If A and B cut each other at C and D,
   then $AC + BC$ = the sum of the two radii.
   But $AC + BC$ is greater than AB, *I.* 20
   which is impossible.
   Hence the two circles do not intersect.

   Nor can each lie entirely outside the other without touching, for then the sum of the radii would be less than AB.
   Hence the circles must touch externally.

2. Let A and B be the centres of two circles, and let AB be = the difference of the two radii.

   If A and B cut each other at C and D,
   then $AC - BC$ = the difference of the two radii.
   But $AC - BC$ is less than AB, *I.* 20, *Cor.*
   which is impossible.
   Hence the two circles do not intersect.

   Nor can one lie entirely inside the other without touching, for then the difference of the radii would be greater than AB.
   Hence the circles must touch internally.

PROPOSITION 14.

1. Since the chords are all equal, the perpendiculars drawn to them from the centre are all equal. *III.* 14
   But these perpendiculars are drawn to their middle points. *III.* 3

   $\therefore$ the distances of the middle points of the chords from the centre are equal;

   $\therefore$ the middle points of the chords lie on the $\bigcirc^{ce}$ of a circle.

2. Let AB be 8 inches and CD 6 inches.
   Find O the centre of the circle;

draw $OEF \perp AB$ and CD, meeting AB in E and CD in F; and join OA, OC.

Then $AE = 4$ inches, and $CF = 3$ inches; *III.* 3

$\therefore$ the right-angled triangle OEA has $OA = 5$ inches and $AE = 4$ inches; $\therefore OE = 3$ inches. *I.* 47, *Cor.*

Similarly the right-angled triangle OFC has $OC = 5$ inches and $CF = 3$ inches;

$\therefore OF = 4$ inches (I. 47, Cor.); $\therefore EF = OF - OE = 1$ inch.

3. Let AC, BD (fig. to III. 35) intersect at E, and make $\angle AEF = \angle DEF$; F being the centre of the circle.

From F draw $FG \perp AC$, and $FH \perp BD$.

Then the right-angled $\triangle$s FEG, FEH are congruent, and $FG = FH$, *I.* 26

$\therefore AC = BD$. *III.* 14

4. Repeat the proof of the preceding deduction with reference to the fig. to III. 35, Cor.

5. Let AB (fig. to III. 14) be a chord of fixed length.

Find E the centre, draw $EF \perp AB$, and join EA.

Then $EF^2 = EA^2 - AF^2$. *I.* 47, *Cor.*

Now EA^2 is fixed, since EA, the radius of the circle, is always the same; and AF^2 is fixed, since AF is half of AB (*III.* 3) a fixed length;

$\therefore EF^2$ and consequently EF is fixed.

6. The converse to be proved is (fig. to III. 14) that if EF is fixed, AB is fixed.

Now $AF^2 = AE^2 - EF^2$, *I.* 47, *Cor.*

and AE^2 and EF^2 are fixed;

$\therefore AF^2$, and consequently AF, and its double AB, are fixed.

7. Let AC and BD (figs. to III. 35 and III. 35, Cor.) be equal chords: to prove $AE = DE$, and $CE = BE$.

Find F the centre of the circle, draw $FG \perp AC$, $FH \perp BD$, and join FE.

Since $AC = BD$, $AG = DH$, *III.* 3

and $FG = FH$; *III.* 14

$\therefore$ in the right-angled $\triangle$s FGE, FHE, $GE = HE$, by the eleventh deduction from I. 47;

$\therefore AG + GE = DH + HE$, or $AE = DE$.

Now since $AC = BD$,

$\therefore AC - AE = BD - DE$ in the first figure,

and $AE - AC = DE - BD$ in the second figure;

$\therefore \quad CE = BE$.

PROPOSITION 15.

1. Let P be a given point within a circle whose centre is O, and let the chord AB drawn through P be $\perp OP$.

   Draw CD any other chord through P, and from O draw $OE \perp CD$.

   Then OP is greater than OE, by the first deduction from I. 19

   $\therefore AB$ is less than CD, *III.* 15

   and CD is any chord through P.

2. Let AC, BD (fig. to III. 35) intersect at E, and let $\angle FEA$ be greater than $\angle FED$: to prove BD greater than AC.

   If a chord $A'EC'$ were drawn on the other side of FE from AEC, and making with FE an angle equal to that made by AEC, it would be equal to AEC, by the third deduction from III. 14.

   and it would be farther from the centre than BED, because $\angle FEA'$ is greater than $\angle FED$;

   $\therefore FH$ is less than the perpendicular from F on $A'EC'$, that is, FH is less than FG;

   $\therefore BD$ is greater than AC. *III.* 15

3. Repeat the preceding proof with reference to fig. to III. 35, Cor., and quote the fourth deduction from III. 14.

4. Let the two circles whose centres are G and H cut each other at A and B.

   Through A draw $CAD \parallel GH$ and meeting the two $\bigcirc^{ces}$ at C and D. CD is the greatest straight line.

   For through A draw any other straight line EAF meeting the $\bigcirc^{ces}$ at E and F. From G and H draw GK, HL $\perp CAD$, and GM, $HN \perp EAF$; from H draw $HP \perp GM$.

   Then it may be proved, as in the eighth deduction from III. 3, that $CD = 2\ KL = 2\ GH$, and that $EF = 2\ MN = 2\ PH$.

   Now GH is greater than PH; *I.* 19, *Cor.*

   $\therefore CD$ is greater than EF.

   If round A as a pivot the straight line EAF be supposed to revolve clockwise, the point E will move along the $\bigcirc^{ce}$ of the circle G till it coincides with A; if the revolution be continued, the point E will move round the $\bigcirc^{ce}$ towards B, and the straight line EAF will now be AEF. When this is the

case, EF will $= 2\,AN - 2\,AM$, instead of as before $2\,AN + 2\,AM$. But $2\,AN - 2\,AM = 2\,MN = 2\,PH$; and GH is greater than PH. As the point E moves along the $\bigcirc^{ce}$ towards B, it will be seen that the angle which GP makes with GH becomes smaller and smaller, and consequently that PH becomes smaller and smaller. Now EF is always $= 2\,PH$; $\therefore$ EF becomes smaller and smaller. When E coincides with B, F also coincides with B, so that the direction of the evanescent EF is that of the common chord AB, which is $\perp$ GH, by the tenth deduction from I. 8.

PROPOSITION 16.

1. See III. 17.
2. For only one perpendicular can be drawn to a radius at its extremity.
3. The straight line joining the centres of the two (or more) circles passes through their point of contact; *III.* 11, 12
   $\therefore$ the perpendicular to that straight line at the point of contact will be a tangent to both circles. *III.* 16
4. Let AB be one of the equal chords, C its middle point, and O the centre of the given circle.
   Join OC.
   Then AB is $\perp$ OC. *III.* 3
   But OC is a radius of the circle on which the middle points of the equal chords lie, by the first deduction from III. 14;
   $\therefore$ AB is a tangent to this circle.
5. Since the distance of the foot of the perpendicular from the centre on the straight line is less than a radius, the foot of the perpendicular must be inside the circle; *III. Def.* 1, *Cor.* 2
   $\therefore$ the straight line must cut the circle.
   Since the distance of the foot of the perpendicular from the centre on the straight line = a radius, the foot of the perpendicular must be on the $\bigcirc^{ce}$ of the circle;
   $\therefore$ the straight line must touch the circle. *III.* 16
   Since the distance of the foot of the perpendicular from the centre on the straight line is greater than a radius, the foot of the perpendicular must be outside the circle. *III. Def.* 1, *Cor.* 2

and every other point in the straight line is farther distant from the centre than the foot of the perpendicular; *I.* 19, *Cor.*
$\therefore$ the straight line must lie outside the circle.

6. From the centre of the given circle draw a straight line $\perp$ the given straight line, and at the points (two) where this straight line cuts the circle draw tangents.
 The proof follows from I. 28.

7. From the centre of the given circle draw a straight line $\parallel$ the given straight line, and at the points (two) where this straight line cuts the circle draw tangents.
 The proof follows from I. 29.

8. Take any point in the given straight line, and at that point draw another straight line making the given angle with the given straight line. *I.* 23
 Employ the sixth deduction to draw two tangents $\parallel$ the second straight line.
 The proof follows from I. 29.
 If it be considered that at the point assumed in the given straight line, there may be drawn two straight lines making with it the given angle, then there may be four tangents.

PROPOSITION 17.

1. (*a*) $AB^2 + EB^2 = AE^2 = AC^2 + EC^2$. *I.* 47
 But $EB^2 = EC^2$; $\therefore$ $AB^2 = AC^2$, and $AB = AC$.
 (*b*) Join BC.
 Because $EB = EC$, $\therefore$ $\angle EBC = \angle ECB$. *I.* 5
 But $\angle EBA = \angle ECA$;
 $\therefore$ $\angle ABC = \angle ACB$; $\therefore$ $AB = AC$; *I.* 6

2. Triangles ABE, ACE are congruent, by the previous deduction and I. 8;
 $\therefore$ $\angle BAE = \angle CAE$.

3. Of the four angles of the quadrilateral $ABEC$, two, $\angle ABE$ and $\angle ACE$, are right;
 $\therefore$ $\angle BAC + \angle BEC = 2$ rt. $\angle$s.
 The angle between two tangents to a circle is supplementary to the angle between the radii drawn to their points of contact.

4. Let BDC be the circle (fig. to III. 17), and A the external point.
   The points of contact of the tangents are determined by joining the centre E to F and G, the points where a perpendicular to AE through D meets the circle AFG.
   Now since this perpendicular meets the circle AFG in only two points, there can be only two tangents.
5. Let $EFGH$ (fig. to IV. 7) be any quadrilateral circumscribed about a circle, and let the points of contact of the sides be A, B, C, D.
   Then $EA = ED, FA = FB;$ *III.* 17, *Cor.*
   $\therefore$ $EF = ED + FB.$
   Similarly $HG = HD + GB;$
   $\therefore$ $EF + HG = EH + FG.$
6. If a polygon of an even number of sides is circumscribed about a circle, the sum of the first, third, fifth, &c. sides is equal to the sum of the second, fourth, sixth, &c. sides.
7. Let $EFGH$ (fig. to IV. 7) be a $\|^{m}$ circumscribed about a circle.
   Then by the fifth deduction $EF + HG = EH + FG$.
   But $EF = HG$, and $EH = FG;$ *I.* 34
   $\therefore$ twice EF = twice EH, and $EF = EH$.
   Hence all the sides of $EFGH$ are equal, and the $\|^{m}$ is a rhombus.
8. Since $AB = AC$, and $EB = EC$, this deduction is merely a statement in other words of the seventh deduction from I. 8.

PROPOSITION 18.

1. For they are both $\perp$ the diameter.
2. Let AB, CD, two chords in the greater of two concentric circles, be tangents at the points E, F to the smaller circle.
   Find O the common centre, and join OE, OF.
   Then OE is $\perp AB$ and $OF \perp CD$. *III.* 18
   But $OE = OF;$ $\therefore AB = CD$. *III.* 14
3. Since OE (fig. to previous deduction) is $\perp AB$,
   $\therefore$ it bisects AB. *III.* 3
4. Let P be the point inside the circle, M the given length.
   Take any point A on the $\bigcirc^{ce}$ of the circle, and with A as centre and M as radius cut the circle at B; join AB, and from O, the centre of the circle, draw $OE \perp AB$.

With O as centre and OE as radius describe a circle; from P draw a tangent to the inner circle, cutting the outer circle at C and D. CD is the chord required.
For by the second deduction $CD = AB = M$.

The given point may be outside the given circle; when it is, the given length may be as small as we please, but not greater than the diameter of the given circle. When the given point is inside the given circle, the given length may not be less than the shortest chord that can be drawn through the given point (see the first deduction from III. 15), nor greater than the diameter.

5. Let ABC be a circle, and let DE be a secant cutting it at the points B and C.

   Find F the centre, and join FB, FC.

   Then $\triangle\ FBC$ is isosceles; $\therefore\ \angle\ FBE = \angle\ FCD$. *I.* 5
   Suppose DE to move gradually away from the centre; then the points B, C and also the straight lines FB, FC will approach one another and ultimately coincide.
   When FB, FC coincide, the secant DE becomes a tangent, and $\angle$s FBE, FCD, which remain always equal, are adjacent, and consequently right.

6. Angles ACD, BDC are right; $\therefore\ AC$ is $\parallel$ BD. *I.* 27 or 28

PROPOSITION 19.

1. No.
2. The point of contact C may be considered as the middle of an infinitely small chord in the circle.
3. In the straight line drawn through the given point $\perp$ the given straight line.
4. The problem is generally impossible. It is possible when the perpendiculars to the two given straight lines at the two given points meet on the bisector of the angle contained by the straight lines.
5. Let BC (fig. to III. 17), the chord of contact, cut AE at H.

   Then $\triangle\ ABE$ is right-angled (III. 19), and so is $\triangle\ AHB$, by the eighth deduction from III. 17;
   $\therefore\ \angle\ EBH = \angle\ BAH$, by the seventeenth deduction from I. 32.
   But $\angle\ BAH$ = half of $\angle\ BAC$;
   $\therefore\ \angle\ EBH$ = half of $\angle\ BAC$.

PROPOSITION 20.

1. No.
2. In the figures to the proposition, if BE and EC be in one straight line, $\angle\ BEC$ is a straight angle, and BAC is a semicircle;

   $\therefore\ \angle\ BAC =$ half of $\angle\ BEC =$ a right angle.
3. Since the points B and C are fixed, and since the centre E is fixed, $\angle\ BEC$ is a constant magnitude.

   Now wherever A is situated on the arc BAC, $\angle\ BAC =$ half of $\angle\ BEC$;

   $\therefore\ \angle\ BAC$ is constant in magnitude.

PROPOSITION 21.

1. For $\angle\ AEB = \angle\ DEC$ (I. 15), and $\angle\ BAE = \angle\ CDE$; *III.* 21

   $\therefore$ the triangles are mutually equiangular. *I.* 32, *Cor.* 1
2. Repeat the preceding proof with reference to the figure to III. 35, Cor., but omit the quotation I. 15.
3. Let A, B, C be the three points.

   Join AB, AC, and from B draw any straight line BD on the same side of BC as A; from C draw CE, making with BD an angle $CEB = \angle\ CAB$, by the second deduction from I. 31.

   Then E is a fourth point on the $\bigcirc^{ce}$.
4. By the third deduction from III. 20, $\angle\ D$ is constant; and $\angle\ A$ is constant;

   $\therefore\ \angle\ A + \angle\ D$ is constant.

   Now $\angle\ A + \angle\ D + \angle\ ABD + \angle\ ACD = 4$ rt. $\angle$s;

   $\therefore\ \angle\ ABD + \angle\ ACD$ is constant.
5. Yes. Because the reflex angle BDC may be shown to be constant; and the sum of the four angles of the concave quadrilateral $ABDC = 4$ rt. $\angle$s.
6. Join AD.

   Then $\angle\ ACB = \angle\ ADC + \angle\ DAC$.

   Now $\angle\ ACB$ and $\angle\ ADC$ are each invariable in magnitude, by the third deduction from III. 20;

   $\therefore\ \angle\ DAC$ or $\angle\ DAE$ is invariable in magnitude;

   $\therefore$ arc DE is invariable.

PROPOSITION 22.

1. For the opposite angles are both equal and supplementary;
   $\therefore$ they must be right.
2. For $\angle\ AED$ (fig. to III. 35, Cor.) $= \angle\ BEC$, and $\angle\ EBC$ $= \angle\ EAD$. *III. 22, Cor.*
3. Let $ABCDEF$ (fig. to IV. 15) be a hexagon inscribed in a circle. Join BE.
   Since $ABEF$ is a quadrilateral inscribed in a circle,
   $\angle\ FAB + \angle\ BEF = 2$ rt. $\angle$s.
   Similarly $\angle\ BCD + \angle\ BED = 2$ rt. $\angle$s;
   $\therefore\ \angle\ FAB + \angle\ BCD + \angle\ DEF = 4$ rt. $\angle$s.
   Now the sum of all the angles of the hexagon $= 8$ rt. $\angle$s. *I. 32, Cor. 3*
4. Let ACB be an arc of a circle on the chord AB, and let C be any point in the arc. In the arcs AC and CB any two points D and E are taken, and it is to be proved that the sum of the angles in the segments ADC, BEC is constant.
   Join AC, BC, AD, CD, BE, CE.
   Since $ABCD$ is a quadrilateral inscribed in a circle,
   $\angle\ ABC + \angle\ ADC = 2$ rt. $\angle$s.
   Similarly $\angle\ BAC + \angle\ BEC = 2$ rt. $\angle$s;
   $\therefore\ \angle\ ABC + \angle\ BAC + \angle\ ADC + \angle\ BEC = 4$ rt. $\angle$s.
   Now though $\angle$s ABC, BAC are not fixed in magnitude, their sum is fixed, since it differs from 2 rt. $\angle$s by $\angle\ ACB$ which is fixed;
   $\therefore\ \angle\ ADC + \angle\ BEC$ is fixed.
5. (*a*) At O, the centre of the circle, make an angle $AOB =$ one-third of 4 rt. $\angle$s;
   then $\angle\ ACB$ at the $\bigcirc^{ce}$ standing on the same arc $=$ one-third of 2 rt. $\angle$s;
   $\therefore\ \angle\ ADB$ in the conjugate segment $=$ two-thirds of 2 rt. $\angle$s.
   (*b*) At O, the centre of the circle, make an angle $AOB =$ one-fourth of 4 rt. $\angle$s;
   then $\angle\ ACB$ at the $\bigcirc^{ce}$ standing on the same arc $=$ one-fourth of 2 rt. $\angle$s;
   $\therefore\ \angle\ ADB$ in the conjugate segment $=$ three-fourths of 2 rt. $\angle$s.
   (*c*) At O, the centre of the circle, make an angle $AOB =$ one-sixth of 4 rt. $\angle$s;

then $\angle ACB$ at the $\bigcirc^{ce}$ standing on the same arc = one-sixth of 2 rt. $\angle$s;

$\therefore$ $\angle ADB$ in the conjugate segment = five-sixths of 2 rt. $\angle$s.

(*d*) At O, the centre of the circle, make an angle AOB = one-eighth of 4 rt. $\angle$s;

then $\angle ACB$ at the $\bigcirc^{ce}$ standing on the same arc = one-eighth of 2 rt. $\angle$s;

$\therefore$ $\angle ADB$ in the conjugate segment = seven-eighths of 2 rt. $\angle$s.

6. Because $\angle$s ACB, AOB are right,

$\therefore$ the points A, C, B, O are concyclic;

$\therefore$ $\angle OAB = \angle OCB$, and $\angle OBA = \angle OCA$. *III.* 21

Now $\angle OAB = \angle OBA$; $\therefore$ $\angle OCB = \angle OCA$;

$\therefore$ OC bisects $\angle ACB$.

7. Because $\angle$s ACB, AOB are right,

$\therefore$ the points A, C, B, O are concyclic;

$\therefore$ $\angle OAB$ is supplementary to $\angle OCB$, *III.* 22

and $\angle OBA = \angle OCA$. *III.* 21

Now $\angle OAB = \angle OBA$;

$\therefore$ $\angle OCB$ is supplementary to $\angle OCA$;

$\therefore$ if BC be produced to B', OC bisects $\angle ACB'$.

8. Let ABC, DEF (fig. to III. 26) be two circles, and let the chords BC, EF cut off the similar segments BAC, EDF: the segments conjugate to these, BKC, ELF, will also be similar.

Because segment BAC is similar to segment EDF,

$\therefore$ $\angle A = \angle D$;

$\therefore$ the angle in segment BKC which is supplementary to $\angle A$ = the angle in segment ELF which is supplementary to $\angle D$;

$\therefore$ segment BKC is similar to segment ELF.

9. Let A, B, C be the three points.

Join AB, AC, and from B draw any straight line BD on the opposite side of BC from A; from C draw CE making with BD an angle CEB = the supplement of $\angle CAB$, by the second deduction from I. 31.

Then E is a fourth point on the $\bigcirc^{ce}$.

10. A, Z, O, Y are concyclic, because $\angle$s AZO, AYO are each right, and therefore supplementary. *III.* 22

B, X, O, Z are concyclic, because $\angle$s BXO, BZO are each right, and therefore supplementary. *III.* 22

C, Y, O, X are concyclic, because $\angle$s CYO, CXO are each right, and therefore supplementary. *III.* 22

A, B, X, Y are concyclic, because $\angle$s AXB, AYB are each right, and therefore equal. *III.* 21
B, C, Y, Z are concyclic, because $\angle$s BYC, BZC are each right, and therefore equal. *III.* 21
C, A, Z, X are concyclic, because $\angle$s CZA, CXA are each right, and therefore equal. *III.* 21

PROPOSITION 23.

1. Let ACB, ADB be two segments of circles on the same side of the same chord AB, and let ACB be greater than ADB.
   Then ADB must fall entirely within ACB. *III.* 10
   Make the same construction as in the proposition, and prove by I. 16 that $\angle$ ACB is less than $\angle$ ADB.
2. Join BD (fig. to III. 21), and let a circle be described through the points C, B, D. *III.* 1, *Cor.* 2
   Then, since $\angle A = \angle C$, segment BAD is similar to segment BCD;
   and they are on the same side of the same chord BD;
   $\therefore$ they must coincide, that is, C lies on the arc BAD.

PROPOSITION 24.

1. For they may be made to coincide; hence the segments conjugate to them, which are also similar to each other (III. 22), must coincide (III. 23); $\therefore$ the circles are equal.
   Or, since the segments may be made to coincide, the two circles will then have more than two points in common, and are therefore coincident. *III.* 10
2. In these circles ABC and ABC' are similar segments, and they are on equal chords AC, AC';
   $\therefore$ these segments are equal;
   $\therefore$ the circles themselves are equal, by the previous deduction.
3. Because $\angle ABE + \angle BEC = 2$ rt. $\angle$s, and $\angle BAD + \angle ADC = 2$ rt. $\angle$s; *I.* 29
   $\therefore \angle ABE + \angle BEC = \angle BAD + \angle ADC$;

$\therefore \angle BEC = \angle ADC = \angle BCE;$ *I.* 29

$\therefore \quad BE = BC;$ *I.* 6

$\therefore$ the circles described about $\triangle$s BCD, BED are equal, by the previous deduction.

PROPOSITION 25.

1. By extending I. 28, Cor. thus: Straight lines which are perpendicular to parallel straight lines are parallel, it will be seen that if DF and EF do not meet they must be parallel; $\therefore AB$ and BC must be parallel, which is absurd.
2. Join EC.
   Then, because $\angle BAE = \angle ABE$, $EA = EB$. *I.* 6
   Now $\triangle$s ADE, CDE are congruent, and $EA = EC$; *I.* 4
   $\therefore EA = EB = EC$, and E is the centre of the circle of which ABC is an arc. *III.* 9
3. This is either III. 1, Cor. 2, or III. 25 expressed in other words. The problem is impossible when the three points are in one straight line.
4. They all pass through the centre of the circle described through the three vertices of the triangle. *III.* 3
5. Let A, B, C, D be the four points (figs. to III. 21 and 22).
   If A and C lie on the same side of BD, and $\angle A$ be $= \angle C$; or if A and C lie on opposite sides of BD, and $\angle A$ be supplementary to $\angle C$; the four points A, B, C, D are concyclic. Hence, if the problem be possible, the required point will be the centre of a circle described through any three of the four points.

PROPOSITION 26.

1. Join BC.
   Then $\angle ABC = \angle BCD$ (I. 29); $\therefore$ arc $AC =$ arc BD.
   Hence also arc AC + arc CD = arc BD + arc CD,
   or arc AD = arc BC.
2. Let ABC, DEF (fig. to III. 27) be two equal circles, and let $\angle BGC$ be greater than $\angle EHF$.
   At G make $\angle BGK = \angle EHF$.

Then arc BK = arc EF. *III.* 26

But arc BC is greater than arc BK;

$\therefore$ arc BC is greater than arc EF.

3. Since the two angles are equal, the arcs on which they stand are equal;

$\therefore$ each arc is a semicircle, and the diagonal a diameter of the circle.

4. This follows from the previous deduction, because when two opposite angles of a quadrilateral inscribed in a circle are equal, each must be a right angle. *III.* 22

5. Let BAC (one of the figs. to III. 27) be a segment containing an acute angle A.

Then the conjugate segment BKC must contain an obtuse angle; *III.* 22

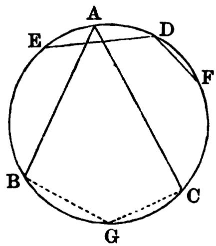
$\therefore$ arc BAC must be greater than arc BKC, by the second deduction.

But arc BAC and arc BKC make up a whole $\bigcirc^{ce}$;

$\therefore$ segment BAC must be greater than a semicircle, and segment BKC less.

6. Let $\angle BAC + \angle EDF = 2$ rt. $\angle$s.

In arc BC take any point G, and join BG, CG.



Then $\angle BAC + \angle BGC = 2$ rt. $\angle$s; *III.* 22

$\therefore \angle BGC = \angle EDF$;

$\therefore$ arc BAC = arc $EBCF$. *III.* 26

But arc BAC + arc BGC = the whole $\bigcirc^{ce}$;

$\therefore$ arc $EBCF$ + arc BGC = the whole $\bigcirc^{ce}$.

7. Apply the circle ABC to the circle DEF so that the centre G may fall on the centre H, and so that GB may fall along HE; then B will fall on E, because $GB = HE$;

and GC will fall along HF, because $\angle G = \angle H$,

and C will fall on F, because $GC = HF$.

Hence, since the circles coincide, *III. Def.* 1, *Cor.* 3

the arc BKC coincides with the arc ELF;

$\therefore$ arc BKC = arc ELF.

8. Let AC, BD (fig. to III. 35) intersect at E.

Through B draw $BL \parallel CA$.

Then $\angle AED = \angle LBD$. *I.* 29

Now $\angle LBD$ at the $\bigcirc^{ce}$ stands on arc LD, which = arc AD + arc AL = arc AD + arc BC, by the first deduction; hence an angle at the centre equal to $\angle LBD$ or $\angle AED$ stands on an arc equal to half the sum of the arcs AD and BC. *III.* 20

9. Repeat the preceding proof with reference to the fig. to III. 35, Cor., substituting only $-$ for $+$, and 'difference' for 'sum.'
10. Find the centre of the circle, and at it make an angle equal to the third, fourth, sixth, or eighth part of four right angles. The arc subtended by this angle will be the third, fourth, sixth, or eighth part of the $\bigcirc^{ce}$.

PROPOSITION 27.

1. Join BC.
   Then, since arc AC = arc BD, $\angle ABC = \angle BCD$; *III.* 27
   $\therefore AB$ is $\parallel CD$. *I.* 27
2. Let ABC, DEF (fig. to III. 27) be two equal circles, and let arc BC be greater than arc EF.
   Suppose arc BK = arc EF, and join GK.
   Then $\angle BGK = \angle EHF$. *III.* 27
   But $\angle BGC$ is greater than $\angle BGK$;
   $\therefore \angle BGC$ is greater than $\angle EHF$.
3. Let CE (fig. to III. 3) be a diameter of the circle;
   then the arc CAE = the arc CBE; $\therefore \angle CAE = \angle CBE$.
   But $\angle CAE + \angle CBE$ = 2 rt. $\angle$s; *III.* 22
   $\therefore \angle CAE$ or $\angle CBE$ is a right angle.
4. Let BAC (fig. to III. 27) be a segment greater than a semicircle;
   then BKC must be a segment less than a semicircle;
   $\therefore \angle BAC$ must be less than $\angle BKC$, by the second deduction.
   Now $\angle BAC + \angle BKC$ = 2 rt. $\angle$s; *III.* 22
   $\therefore \angle BAC$ must be acute, and $\angle BKC$ obtuse.
5. Let arcs BGC and $EBCF$ (fig. to the sixth deduction from III. 26) be equal to the whole $\bigcirc^{ce}$: to prove $\angle BAC$ supplementary to $\angle EDF$.
   Since arc BGC + arc $EBCF$ = the $\bigcirc^{ce}$ = arc BGC + arc BAC;

$\therefore$ arc $EBCF$ = arc BAC; $\therefore$ $\angle\ EDF = \angle\ BGC$. *III.* 27
Now $\angle\ BAC$ is supplementary to $\angle\ BGC$;
$\therefore$ $\angle\ BAC$ is supplementary to $\angle\ EDF$.

6. Apply the circle ABC to the circle DEF so that the centre G may fall on the centre H, and so that GB may fall along HE;
then B will fall on E, because $GB = HE$,
and the arc BKC will fall on the arc EF, because the circles coincide. *III. Def.* 1, *Cor.* 3
Now since arc BC = arc EF,
the point C will coincide with F, and consequently GC with HF;
$\therefore$ $\angle\ BGC = \angle\ EHF$.

7. Let two circles whose centres are E and F touch each other internally at C, E being the centre of the inner circle. Let AB, a chord of the circle whose centre is F, touch the circle whose centre is E at D: to prove $\angle\ ACD = \angle\ BCD$.

Join CD, produce it to meet the large circle at G, and join ED, FG. Draw also the straight line CEF. *III.* 11
Then $\angle\ ECD = \angle\ EDC$, and $\angle\ FCD = \angle\ FGC$; *I.* 5
$\therefore$ $\angle\ EDC = \angle\ FGC$, and FG is $\parallel$ ED. *I.* 28
Now ED is $\perp$ AB; *III.* 18
$\therefore$ FG is $\perp$ AB, and bisects AB; *III.* 3
$\therefore$ FG bisects $\angle\ AFB$, by the third deduction from I. 26.
Hence arc AG = arc BG; *III.* 26
and $\angle\ ACG = \angle\ BCG$. *III.* 27

PROPOSITION 28.

1. Since chord AC = chord BD, $\therefore$ arc AC = arc BD;
$\therefore$ $\angle\ ABC = \angle\ BCD$;
$\therefore$ AB is $\parallel$ CD.

2. Let A be the given point, and BC the given straight line.
With B as centre and BA as radius describe an arc cutting BC at D;
with A as centre and AB as radius describe an arc BE;
with B as centre, and a radius equal to the chord AD, cut the arc BE in F. AF is the straight line required.

3. Let two equal circles cut each other at A and B, and let CD drawn through A meet the circles at C and D: to prove $BC = BD$.

Since AB is a common chord in the two equal circles, it cuts off equal arcs from them.

Now $\angle BCA$ stands on the one arc, and $\angle BDA$ on the other.

$\therefore \angle BCA = \angle BDA$;

$\therefore \quad BC = BD$. *I.* 6

PROPOSITION 29.

1. Since arc AC = arc BD,

$\therefore$ arc AC + arc CD = arc BD + arc CD;

$\therefore$ arc AD = arc BC; $\therefore$ chord AD = chord BC.

2. Apply the circle ABC to the circle DEF, so that their centres may coincide; then the $\bigcirc^{ces}$ will also coincide. Let the $\bigcirc^{ce}$ ABC be so placed that B coincides with E, and arc BGC falls along arc EHF;

then C will coincide with F, because arc BGC = arc EHF;

$\therefore$ BC will coincide with EF; $\therefore$ $BC = EF$.

PROPOSITION 30.

1. The straight line joining their centres bisects perpendicularly their common chord, by the tenth deduction from I. 8.

$\therefore$ it bisects all the four arcs cut off by this common chord.

2. If a diameter of a circle is $\perp$ a chord, it bisects the chord, and consequently bisects the arcs cut off by the chord.

3. The distance AB may be bisected perpendicularly without AB being joined.

With A as centre and AB as radius describe a circle.

with B as centre and BA as radius describe a circle cutting the former one in two points.

The straight line joining these two points will bisect AB perpendicularly, and consequently will bisect the arc ADB.

4. Let AEB be any other triangle on the base AB, and having its vertex E on the arc ADB.

From E draw $EF \perp AB$.

Then EF is less than EC, *I.* 19, *Cor.*

and consequently than DC.

Hence the altitude of $\triangle\ EAB$ is less than that of $\triangle\ DAB$;

$\therefore\ \triangle\ EAB$ is less than $\triangle\ DAB$.

PROPOSITION 31.

1. Let ABC be an isosceles triangle, having $AB = AC$.

   Take D the middle point of the base BC, and join AD.

   Then $\angle$s ADB, ADC are right, by the first deduction from I. 8.

   $\therefore$ the circles described on AB and AC as diameters will pass through D.

2. Let ABC be a triangle, and let AD be $\perp$ BC or BC produced.

   Since $\angle$s ADB, ADC are right, the circles described on AB and AC as diameters will pass through D.

3. To solve I. 11. Let AB be the given straight line, C the given point in it.

   Take any point D not in AB, and with D as centre and DC as radius describe a circle cutting AB at E; join ED, and produce it to meet the circle at F; and join CF.

   Then FCE is a semicircle; $\therefore\ \angle\ FCE$ is right.

   To solve I. 12. Let AB be the given straight line, C the given point outside it.

   Take any point D in AB; join CD, and on it as diameter describe a semicircle CED cutting AB in E; join CE.

   Then $\angle\ CED$ is right.

4. Take any point A on the $\bigcirc^{ce}$ of the circle, and with the set square draw AB, $AC \perp$ each other, and cutting the circle in B and C.

   Then BC is a diameter.

   Take another point on the $\bigcirc^{ce}$, and draw another diameter.

   The two diameters will intersect at the centre.

5. For $\angle$s ABE, ACE, being in semicircles, are right;

   $\therefore$ AB and AC are tangents.

6. Let the second circle pass through O the centre of the first, and intersect the first at the points A, B.

   Then A, O, B, being three points on the $\bigcirc^{ce}$ of a circle, will be the vertices of a triangle;

$\therefore \angle AOB$ is less than a straight angle;

$\therefore$ the exterior segment of the first circle, whose base is AB, is greater than a semicircle;

$\therefore$ the angle in it is acute.

7. AB is the radius of a circle; on AB as diameter a circle ADB is described, and from B there is drawn BC a chord of the first circle, cutting the second circle at D. Join AD.

Then $\angle ADB$ being in a semicircle is right;

$\therefore AD$ bisects BC. *III.* 3

8. Let two circles intersect at A and B; through A draw AC a diameter of the one, and AD a diameter of the other; to prove C, B, D in one straight line.

Join BA, BC, BD.

Then $\angle$s ABC, ABD are right;

$\therefore CB$ and BD are in one straight line. *I.* 14

9. Let AB and CD be sides of the two given squares, AB being the greater.

On AB as diameter describe a semicircle, and with B as centre and CD as radius cut the semicircle at E.

AE is a side of the square required.

For $\triangle AEB$ is right-angled;

$\therefore AE^2 = AB^2 - BE^2 = AB^2 - CD^2$.

10. Let ABC be a triangle (fig. to III. 31) right-angled at A, and let E be the middle point of BC.

The circle described on BC as diameter will pass through A, and of this circle EA, EB, EC are radii, and therefore equal.

11. If a point in one of the sides of a triangle be equidistant from the three vertices, the triangle is right-angled.

Let ABC be a triangle (fig. to III. 31), and let E be equidistant from A, B, C.

With E as centre, and EA as radius describe a circle which will pass through B and C.

Then BAC is a semicircle; $\therefore \angle BAC$ is right.

12. Let the common tangent at A meet BC in E.

Then $EB = EA = EC$; *III.* 17, *Cor.*

$\therefore \angle BAC$ is right, by the preceding deduction.

PROPOSITION 32.

1. Let AC, BD be two parallel tangents, A and B being the points of contact.

Join AB.

Then $\angle\ CAB =$ the angle in the alternate segment AEB $= \angle\ DBA$.

But $\angle\ CAB + \angle\ DBA = 2$ rt. $\angle$s; *I.* 29

$\therefore$ $\angle$s CAB and DBA are right angles, and the alternate segment AEB is a semicircle, and AB a diameter.

2. Let the two circles ABC, ADE (figs. to III. 12, 11) touch each other externally or internally at A, and let BAD divide the circles into pairs of segments.

At A draw PAQ a common tangent to the two circles.

Then (fig. to III. 12) $\angle\ PAB =$ angle in segment ACB,
and $\angle\ QAD =$ angle in segment AED.

Now $\angle\ PAB = \angle\ QAD$;

$\therefore$ segment ACB is similar to segment AED.

Again (fig. to III. 11) $\angle\ PAB =$ angle in segment ACB,
and $\angle\ PAD =$ angle in segment AED;

$\therefore$ segment ACB is similar to segment AED.

Hence also in both figures the other pairs of segments are similar. *III.* 22

3. Let the two circles ABC, ADE (figs. to III. 12, 11) touch each other externally or internally at A, and let BAD, CAE be drawn through A: to prove $BC \parallel DE$.

It was proved in the preceding deduction that segment ACB is similar to segment AED; $\therefore$ $\angle\ ACB = \angle\ AED$;

$\therefore$ BC is $\parallel$ DE. *I.* 27 or 28

4. Let AB, AC (fig. to III. 17) be two tangents to circle BDC, let BE drawn through the centre meet the $\bigcirc^{ce}$ at H, and let BC, HC be joined.

Then $\angle\ BHC = \angle\ ABC = \angle\ ACB$;

$\therefore$ twice $\angle\ BHC = \angle\ ABC + \angle\ ACB$;

$\therefore$ supplement of twice $\angle\ BHC =$ supplement of $(\angle\ ABC + \angle\ ACB)$;

$\therefore$ $\angle\ BCH + \angle\ CBH - \angle\ BHC = \angle\ BAC$;

Now $\angle\ BCH$ is right; *III.* 31

$\therefore$ $\angle\ BCH - \angle\ BHC = \angle\ CBH$;

$\therefore$ twice $\angle\ CBH = \angle\ BAC$.

5. If any point B (fig. to III. 32) be taken on the $\bigcirc^{ce}$ of a circle ABC, and from it there be drawn two straight lines BD, BF, of which BD cuts the circle, and if $\angle\ DBF$ be equal to the angle in the alternate segment, then BF is a tangent.

From B draw $BA \perp BF$, and meeting the $\bigcirc^{ce}$ at A, and join AD.

Then $\angle\ BAD = \angle\ DBF$; *Hyp.*

$\therefore\ \angle\ BAD + \angle\ ABD = \angle\ ABF$,

$=$ a right angle;

$\therefore\ \angle\ ADB =$ a right angle; *I.* 32

$\therefore\ AB$ is a diameter of the circle;

$\therefore\ BF$ is a tangent to the circle. *III.* 16

6. For $\angle\ DAC = \angle\ DAE$. *III.* 27

Now $\angle\ DAC$ or $\angle\ BAC = \angle\ BCA$; *I.* 5

$\therefore\ \angle\ DAE = \angle\ BCA$;

$\therefore\ AE$ is a tangent, by the preceding deduction.

7. In the first fig. to III. 21 join AC, and produce BD to G.

Then $\angle\ CAD = \angle\ CBG$. *III.* 21

Now if BG rotate clockwise round B, the point D will approach nearer and nearer to B, and ultimately coincide with it. When D coincides with B, BDG will be the tangent at B, and $\angle\ CAD$ will have become $\angle\ CAB$; hence the angle made by BC with the tangent at $B =$ the angle in the alternate segment CAB.

In the first fig. to III. 22, produce CD to G.

Then $\angle\ ADG = \angle\ ABC$. *III.* 22, *Cor.*

Now if CG rotate clockwise round C, the point D will approach nearer and nearer to C, and ultimately coincide with it. When D coincides with C, CDG will be the tangent at C, and $\angle\ ADG$ will have become $\angle\ ACG$; hence the angle made by CA with the tangent at $C =$ the angle in the alternate segment ABC.

PROPOSITION 33.

1. If $\angle\ ABG$ be made $= \angle\ BAE$, then $AG = BG$. *I.* 6

2. Let BC be the given base.

On BC describe a segment of a circle containing an angle equal to the given vertical angle;

with B as centre, and the given side as radius, cut the arc of this segment at A, and join AC.

ABC is the required triangle.

3. On BC, the given base, describe a segment of a circle containing an angle equal to the given vertical angle; from B draw $BD \perp BC$ and = the given altitude; through D draw a parallel to BC, meeting the arc of the segment in A, and join AC. ABC is the required triangle.

4. On BC, the given base, describe a semicircle; with B as centre, and the perpendicular as radius, cut the arc of the semicircle at D; join CD, and produce it if necessary to meet at A the arc of a segment described on BC, and containing an angle equal to the given vertical angle; join AB.

ABC is the required triangle.

5. On BC, the given base, describe a segment of a circle containing an angle equal to half the given vertical angle; with B as centre, and the sum of the sides as radius, cut the arc of the segment at D; join CD, at C make $\angle DCA = \angle BDC$, and let CA meet BD at A. ABC is the required triangle.

For $\angle BAC = \angle ADC + \angle ACD$, *I.* 32

$= $ the given angle;

and $BA + AC = BA + AD = BD$.

6. On BC, the given base, describe a segment of a circle containing an angle equal to a right angle increased by half the given vertical angle; with B as centre, and the difference of the sides as radius, cut the arc of the segment at D; join CD, at C make $\angle DCA =$ the supplement of $\angle BDC$, and let CA meet BD produced at A. ABC is the required triangle.

For $\angle BDC = 1$ rt. $\angle + \frac{1}{2}$ given angle.

$\therefore$ supplement of $\angle BDC = 1$ rt. $\angle - \frac{1}{2}$ given angle;

$\therefore$ $\angle ACD = 1$ rt. $\angle - \frac{1}{2}$ given angle.

Now $\angle BAC = \angle BDC - \angle ACD$, *I.* 32

$= (1$ rt. $\angle + \frac{1}{2}$ given angle$)$

$- (1$ rt. $\angle - \frac{1}{2}$ given angle$)$,

$=$ given angle;

and $BA - AC = BA - AD = BD$.

PROPOSITION 34.

Draw any chord cutting off a segment containing an angle equal to the given angle; through the given point draw a chord equal to the other chord, by the fourth deduction from III. 18.

PROPOSITION 35.

1. Let the circles intersect at A and B, and through O, any point in AB, let there be drawn CD a chord in the one circle, and EF a chord in the other.

   Then $CO \cdot OD = AO \cdot OB = EO \cdot OF$;

   $\therefore$ C, E, D, F are concyclic.

2. By the tenth deduction from III. 22, A, B, X, Y are concyclic;

   $\therefore$ $AO \cdot OX = BO \cdot OY$.

   B, C, Y, Z are concyclic; $\therefore$ $BO \cdot OY = CO \cdot OZ$.

3. Because $\angle$s C and D are right,

   $\therefore$ B, C, D, E are concyclic; $\therefore$ $AB \cdot AD = AC \cdot AE$.

4. Because $\angle\, ADE = \angle\, ACB$,

   $\therefore$ B, C, D, E are concyclic; $\therefore$ $AB \cdot AD = AC \cdot AE$.

5. Join P to O, the centre of the circle, and through P draw the chord $CPD \perp OP$.

   Then $AP \cdot PB = CP \cdot PD = CP^2$.

6. To prove VI. B. See figures thereto.

   In fig. 1, from AB cut off $AF = AC$, join DF, BE.

   Then $\triangle$s FAD, CAD are congruent,

   and $\angle\, AFD = \angle\, ACD$. *I.* 4

   Now $\angle\, BFD$ is supplementary to $\angle\, AFD$; *I.* 13

   $\therefore$ $\angle\, BFD$ is supplementary to $\angle\, ACD$;

   $\therefore$ $\angle\, BFD$ is supplementary to $\angle\, AEB$; *III.* 21

   $\therefore$ the four points B, F, D, E are concyclic; *III.* 22

   $\therefore$ $AB \cdot AF = AE \cdot AD$; *III.* 35, *Cor.*

   $\therefore$ $AB \cdot AC = ED \cdot AD + AD^2$, *II.* 3

   $= BD \cdot DC + AD^2$; *III.* 35

   $\therefore$ $AD^2 = AB \cdot AC - BD \cdot DC$,

   In fig. 2, from BA produced cut off $AF = AC$, join DF, BE.

   Then $\triangle$s FAD, CAD are congruent,

   and $\angle\, AFD = \angle\, ACD$. *I.* 4

Now $\angle ACD = \angle BEA$; *III.* 22, *Cor.*
$\therefore \quad \angle AFD = \angle BEA$;
$\therefore$ the four points B, E, F, D are concyclic; *III.* 21
$\therefore AB \cdot AF = AE \cdot AD$;
$\therefore AB \cdot AC = ED \cdot AD - AD^2$, *II.* 3
$= BD \cdot DC - AD^2$; *III.* 35, *Cor.*
$\therefore \quad AD^2 = BD \cdot DC - AB \cdot AC.$

To prove VI. α. See figures thereto.

Produce BA to F, making $AF = AC$, and EA to G, making $AG = AD$. Join FG, BE.

Since $\angle ABE$ is right, *III.* 31
$\therefore \angle BAE$ is complementary to $\angle AEB$. *I.* 32
Since $\angle ADC$ is right,
$\therefore \angle CAD$ is complementary to $\angle ACD$. *I.* 32
Now $\angle AEB = \angle ACD$; *III.* 21
$\therefore \quad \angle BAE = \angle CAD$;
$\therefore \quad \angle FAG = \angle CAD$. *I.* 15
Hence $\triangle$s FAG, CAD are congruent,
and $\angle FGA = \angle CDA$; *I.* 4
$\therefore \angle FGA$ is right.
But because $\angle$s FGE, FBE are right,
$\therefore$ the four points F, G, B, E are concyclic; *III.* 21
$\therefore BA \cdot AF = EA \cdot AG$; *III.* 35
$\therefore BA \cdot AC = EA \cdot AD.$

PROPOSITION 36.

1. Let the secant ECA pass through F, the centre of the circle, and let EB be the tangent.

   Join FB.

   Then $\angle FBE$ is right. *III.* 18
   Now $AE \cdot EC = FE^2 - FC^2$, *II.* 6
   $= FE^2 - FB^2 = EB^2$. *I.* 47, *Cor.*

2. Let the circles intersect at A and B, and let CD be a common tangent.

   Draw AB the common chord, and let it be produced to meet CD at E.

   Then $EC^2 = EA \cdot EB = ED^2$; $\therefore EC = ED$.

3. Let the circles intersect at A and B, and let E be any point in AB produced.

From E draw two tangents EC, ED to the two circles.

Then $EC^2 = EA \cdot EB = ED^2$; $\therefore EC = ED$.

4. By the tenth deduction from III. 22, B, C, Y, Z are concyclic; $\therefore AC \cdot AY = AB \cdot AZ$.

   C, A, Z, X are concyclic; $\therefore BC \cdot BX = BA \cdot BZ$.

   A, B, X, Y are concyclic; $\therefore CA \cdot CY = CB \cdot CX$.

5. Let F be the given point, EC the given straight line, and E the fixed point in it, M^2 the given square.

   Join EF; find the side of a square $= EF^2 - M^2$, by the third deduction from I. 47.

   With F as centre, and the side of this square as radius, describe a circle cutting EC at the points A and C.

   From E draw EB a tangent to this circle, and join FB.

   Then $M^2 = EF^2 - FB^2 = EB^2 = EA \cdot EC$.

6. If the secant EBD revolve anti-clockwise round E till the points B and D coincide, the two segments EB, ED will then be equal, and the rectangle $EB \cdot ED$, which is always $= AE \cdot EC$, will then become EB^2.

PROPOSITION 37.

1. For if not, let EB meet the circle again at D;

   then $BE \cdot ED = AE \cdot EC = BE^2$, which is impossible.

2. If EB be not a tangent to the circle, on the same side of EF as EB draw EG a tangent to the circle, and join FG.

   Then it may be proved, as in the proposition, that $EG = EB$; and $FG = FB$;

   $\therefore$ on the same base EF, and on the same side of it, there are two $\triangle$s EFG, EFB having their pairs of conterminous sides equal, which is impossible.

3. Let A, B be the given points, CD the given straight line.

   Join AB, and produce it to meet CD in E;

   on CD measure off EF such that $EF^2 = EA \cdot EB$, *II.* 14

   and through A, B, F describe a circle. *III.* 1, *Cor.* 2

   There are in general two solutions to this problem, because EF may be measured off in either direction from E. When AB is $\parallel$ CD there is only one solution; the point F is then the point of intersection of the perpendicular bisector of AB with CD. When A and B are on opposite sides of CD, the problem is impossible.

4. Let A be the given point; BC and BD, which intersect at B, the given straight lines. Since A must be situated in one of the four compartments into which CB and DB produced divide the plane, let it be situated between BC, BD.

Bisect $\angle CBD$ by BM; then the centre of the required circle will be in BM. Now since the required circle passes through A, it must pass also through another point A', found by drawing from A a perpendicular to BM, and producing the perpendicular its own length to A'. The problem is now reduced to that of the preceding deduction, and has consequently two solutions.

5. Let BC and BD, which intersect at B, be the given straight lines, and A the centre of the given circle. Let the given circle be situated between BC and BD.

On the opposite sides of BC, BD from A draw two straight lines parallel to BC, BD, and at a distance from them equal to the radius of the given circle. By the previous deduction describe a circle to touch these two straight lines and to pass through the centre of the given circle. The circle required will be concentric with the circle thus found, and will have its radius less than the radius of the circle thus found by the radius of the given circle. This procedure will give two solutions, since there are two solutions to the preceding problem; in both cases the given circle will be touched externally.

Repeat the previous paragraph, substituting 'on the same sides of BC, BD as A,' for 'on the opposite sides of BC, BD from A,' 'greater' for 'less,' and 'internally' for 'externally.'

6. Let A and B be the given points.

Take any point C on the $\bigcirc^{ce}$ of the given circle, and describe a circle to pass through A, B, C. Let this circle cut the given one at D. Join AB, CD, and let them be produced to meet in E; draw EF tangent to the circle ABC; and through A, B, F describe a circle.

ABF is the circle required.

Since two tangents can be drawn from E to the circle ABC, there are in general two solutions to this problem.

[The methods of solution of the third, fourth, and fifth deduction which are given here are substantially those of Vieta. See his *Apollonius Gallus*. The construction for the sixth

deduction here given appeared in the *Monthly Review* for October 1764.]

7. Let CD produced meet AB at E. Describe, by the third deduction, a circle to pass through C, D and touch AB; and let F be its point of contact with AB.

CFD will be the maximum angle.

Take any other point G in AB, and join CG, DG.

Let DG cut the circle CDF at H, and join CH.

Then $\angle\ CHD$ is greater than $\angle\ CGD$; *I.* 16

and $\angle\ CFD = \angle\ CHD$. *III.* 21

There are two maximum angles, since there are two circles which pass through C, D and touch AB, and consequently two points of contact F and F'.

DEDUCTIONS.

1. Let ABC be the triangle.

On AB and AC describe segments of circles each containing an angle $= 1\frac{1}{3}$ rt. $\angle$, that is, an angle = the exterior angle of an equilateral triangle. *III.* 33

Let the arcs of these segments cut each other within the triangle at O; join AO, BO, CO.

Since $\angle\ AOB = 1\frac{1}{3}$ rt. $\angle$, and $\angle\ AOC = 1\frac{1}{3}$ rt. $\angle$;

$\therefore\quad \angle\ BOC = 1\frac{1}{3}$ rt. $\angle$; *I.* 13, *Cor.* 2

$\therefore$ AB, BC, CA subtend equal angles at O.

The problem is impossible if one of the angles of $\triangle\ ABC$ is greater than the exterior angle of an equilateral triangle.

2. Let the two circles, whose centres are G and H, cut each other at A and B.

Join GH, and on it as diameter describe a semicircle GPH. With H as centre, and radius = half the given length, cut the arc of the semicircle at P; join HP, and through A draw EAF or $AEF \parallel HP$, and meeting the two $\bigcirc^{ces}$ at E and F. When A is between E and F, the sum, and when E is between A and F, the difference, of the two chords is equal to the given length.

Join GP, and produce it to meet EF at M; through H draw $HN \parallel GM$, and meeting EF at N.

Then $\angle$ GPH is right; *III.* 31

$\therefore$ $\angle$s M and N are right; *I.* 29

$\therefore$ $AE = 2\,AM$, and $AF = 2\,AN$. *III.* 3

Hence $EF = 2\,MN$,

$= 2\,PH$, *I.* 34

$=$ the given length.

3. Let A and B be the two given points, and let C be a point on the convex $\bigcirc^{ce}$ of the circle such that the tangent at C makes equal angles with AC and BC.

Take any other point D on the $\bigcirc^{ce}$, and join AD, BD. Let AD and the tangent at C intersect each other in E, and join BE.

Then $AD + BD$ is greater than $AE + BE$. *I.* 21

Now $AE + BE$ is greater than $AC + BC$, as may be inferred from the seventh deduction of Book I;

$\therefore$ $AD + BD$ is greater than $AC + BC$.

4. Let AB be the diameter of the given semicircle, O its centre, and let C be any point in AB. With A as centre and AC as radius, the semicircle CFD is described; with B as centre and BC as radius, the semicircle CGE is described; and FG is a common tangent to these two semicircles: to prove that FG touches the original semicircle.

Draw $OH \perp FG$.

Because AF, OH, BG are $\perp$ FG, *III.* 18

$\therefore$ AF, OH, BG are parallel to each other.

Because $AO = BO$, $\therefore$ $OH = \frac{1}{2}(AF + BG)$, *App. I.* 1, *Cor.* 2

$= \frac{1}{2}(AC + BC)$,

$= AO = BO$;

$\therefore$ H is in the $\bigcirc^{ce}$ of the original semicircle;

and since OH is $\perp$ FG,

$\therefore$ FG touches the original semicircle.

[*Lady's and Gentleman's Diary* for 1841, p 45.]

5. Let BDC (fig. to III. 17) be the given circle, M the given length.

Draw any radius EB, and from B draw $BA \perp EB$ and $= M$. Join AE cutting the $\bigcirc^{ce}$ at D; from D draw $DF \perp ED$ and meeting EB produced at F. FD is the required tangent.

For $\triangle$s FDE, ABE are congruent, and $FD = AB$, *I.* 26

$= M$.

6. Let AT be the tangent from A, and H the middle point of BC. Join HA, HT.

Then $\triangle ATH$ is right; *III.* 18

$\therefore \quad AH^2 = AT^2 + TH^2$, *I.* 47

$= AT^2 + BH^2;$

$\therefore 2AH^2 = 2AT^2 + 2BH^2;$

$\therefore 2AH^2 + 2BH^2 = 2AT^2 + 4BH^2;$

$\therefore \quad AB^2 + AC^2 = 2AT^2 + BC^2$, by App. II. 1, and the ninth deduction from II. 4.

7. See the tenth deduction from III. 22.

Since B, X, O, Z are concyclic,

$\therefore \angle OXZ = \angle OBZ$, *III.* 21

$=$ complement of $\angle BAC$,

$= \angle OCY$,

$= \angle OXY$, *III.* 21

since C, Y, O, X are concyclic.

Hence also $\angle XYZ$ is bisected by BY, and $\angle YZX$ by CZ.

Again since B, C, Y, Z are concyclic,

$\therefore \angle AYZ = \angle ABC$, and $\angle AZY = \angle ACB$; *III.* 22, *Cor.*

$\therefore \triangle AYZ$ is equiangular to $\triangle ABC$.

Hence also $\triangle$s XBZ, XYC are equiangular to $\triangle ABC$.

8. Let AX meet the circumscribed circle at X', and join CX'.

Then $\angle BAX = \angle OCX$, because each is the complement of $\angle ABC$;

and $\angle BAX = \angle XCX'$; *III.* 21

$\therefore \quad \angle OCX = \angle XCX'$.

Hence $\triangle$s CXO, CXX' are congruent, and $XO = XX'$. *I.* 26

Similarly for YO and ZO.

9. From the preceding deduction it may be proved that $\triangle$s BOC, $BX'C$ are congruent;

$\therefore$ the circle circumscribed about $\triangle BOC =$ the circle circumscribed about $\triangle BX'C$.

Now the circle circumscribed about $\triangle BX'C$ is the circle circumscribed about $\triangle ABC$;

$\therefore$ the circle circumscribed about $\triangle ABC =$ the circle circumscribed about $\triangle BOC$.

Similarly for $\triangle$s AOB, COA.

10. Let the circles circumscribed about $\triangle$s BFD, CDE cut each other at O. Join OD, OE, OF.

Then $\angle DOF =$ supplement of $\angle B$, *III.* 22

and $\angle DOE =$ supplement of $\angle C$; *III.* 22

$\therefore \quad \angle EOF = \angle B + \angle C;$

$\therefore \angle A + \angle EOF = \angle A + \angle B + \angle C = 2$ rt. $\angle$s;
$\therefore$ the points A, E, O, F are concyclic. *III.* 22

11. Let ABC be any triangle, let D, E, F be vertices of the equilateral triangles described on BC, CA, AB respectively, and let G, H, K be the centres of the circles circumscribed about $\triangle$s DBC, ECA, FAB: to prove $\triangle$ GHK equilateral.

Let the circles circumscribed about $\triangle$s ECA, FAB intersect at O; join OA, OB, OC.

Since $\angle E = \frac{1}{3}$ of 2 rt. $\angle$s;
$\therefore$ $\angle AOC = \frac{2}{3}$ of 2 rt. $\angle$s. *III.* 22
Similarly $\angle AOB = \frac{2}{3}$ of 2 rt. $\angle$s;
$\therefore$ $\angle BOC = \frac{2}{3}$ of 2 rt. $\angle$s;
$\therefore \angle D + \angle BOC = 2$ rt. $\angle$s;
$\therefore$ the points D, B, O, C are concyclic. *III.* 22
Hence the three circles pass through the same point O.

Again KH is $\perp$ AO and KG is $\perp$ BO, by the tenth deduction from I. 8; and $\angle AOB = \frac{2}{3}$ of 2 rt. $\angle$s;
$\therefore$ $\angle K = \frac{1}{3}$ of 2 rt. $\angle$s. *I.* 32, *Cor.* 2
Similarly $\angle G$ and $\angle H$ are each $= \frac{1}{3}$ of 2 rt. $\angle$s;
$\therefore$ $\angle GHK$ is equilateral. *I.* 6, *Cor.*

12. Let BC be the given base, $\angle M$ the given vertical angle.
On BC describe a segment of a circle containing an angle equal to $\angle M$; *III.* 33
from B draw $BD \perp BC$, and equal to the given perpendicular.
Through D draw $DAA' \parallel BC$, and cutting the arc of the segment at A, A'; join AB, AC, or $A'B$, $A'C$.
ABC or $A'BC$ is the required triangle.

13. Let BC be the given base, $\angle M$ the given vertical angle.
On BC describe a segment of a circle containing an angle equal to $\angle M$; *III.* 33
bisect BC at H; with H as centre, and radius = the given median, describe a circle cutting the arc of the segment at A, A'; join AB, AC, or $A'B$, $A'C$.
ABC or $A'BC$ is the required triangle.

14. Let BC be the given base, $\angle M$ the given vertical angle.
On BC describe a segment of a circle containing an angle equal to $\angle M$; *III.* 33
at X, the projection of the vertex on BC, draw $XA \perp BC$, and meeting the arc of the segment at A; join AB, AC.
ABC is the required triangle.

15. Let BC be the given base, $\angle M$ the given vertical angle.

On BC describe a segment of a circle containing an angle equal to $\angle M$; and complete the circle. *III.* 33

Find D the middle of the arc conjugate to the arc of the segment; *III.* 30

join D to N, the point where the bisector of the vertical angle meets the base, and produce it to meet the arc of the segment at A; join AB, AC. ABC is the required triangle.

For $\angle BAN = \angle CAN$. *III.* 27

16. Let BC be the given base, $\angle M$ the given vertical angle.

On BC describe a segment of a circle containing an angle equal to half of $\angle M$; *III.* 33

with B as centre, and radius = the given sum of sides, describe a circle cutting the arc of the segment at D; join BD, and let BD cut at A the arc of a segment described on BC and containing an angle $= \angle M$; join CA, CD.

ABC is the required triangle.

For $\angle BAC = \angle BDC + \angle ACD$; *I.* 32

$\therefore \quad \angle M = \frac{1}{2} \angle M + \angle ACD$;

$\therefore \quad \angle ACD = \frac{1}{2} \angle M = \angle ADC$;

$\therefore \quad AC = AD$, *I.* 6

and $BA + AC = BD$.

Again, on BC describe a segment of a circle containing an angle = a right angle + half of $\angle M$; *III.* 33

with B as centre, and radius = the given difference of sides, describe a circle cutting the arc of the segment at D; join BD, and let BD produced cut at A the arc of a segment described on BC, and containing an angle $= \angle M$; join CA, CD. ABC is the required triangle.

For $\angle BAC = \angle BDC - \angle ACD$; *I.* 32

$\therefore \quad \angle M = 1 \text{ rt. } \angle + \frac{1}{2} \angle M - \angle ACD$;

$\therefore \quad \angle ACD = 1 \text{ rt. } \angle - \frac{1}{2} \angle M = \angle ADC$; *I.* 13

$\therefore \quad AC = AD$, *I.* 6

and $BA - AC = BD$.

17. Let XYZ be the given orthocentric triangle.

Bisect $\angle$s X, Y, Z by XO, YO, ZO intersecting at O.

At X, Y, Z draw BXC, CYA, AZB respectively $\perp$ XO, YO, ZO, and intersecting each other at the points A, B, C.

ABC is the required triangle.

Compare the seventh deduction.

18. Let A and B be the centres of two circles, and let the circle A be greater than the circle B.

With centre A, and radius equal to the difference of the radii of the given circles, describe a circle;
and from B draw BC and BD tangents to this circle.
Join AC, AD, and let them meet the given circle A at E and G;
through B draw $BF \parallel CA$ and $BH \parallel AD$, meeting the $\bigcirc^{ce}$ at F and H; and join EF, GH.

Because $AC = AE - BF$, and $AC = AE - CE$;
$\therefore$ $BF = CE$.
And BF is $\parallel CE$; $\therefore$ $BCEF$ is a $\parallel^{m}$. *I.* 33
Because $\angle ACB$ is right, *III.* 18
$\therefore$ $\angle CEF$ is right, and $\angle BFE$ is right; *I.* 29
$\therefore$ EF is a tangent to both circles.
Similarly GH is a tangent to both circles.

Again, let A and B be the centres of the given circles.

With centre A, and radius equal to the sum of the radii of the given circles, describe a circle. The rest of the construction and the proof are as above, with this exception, that the two minuses must be replaced by two pluses.

The tangents EF and GH in the first case are called direct common tangents; the tangents EF and GH in the second case are called transverse common tangents.

When the two circles touch externally, the two transverse common tangents become coincident, and there are then only three common tangents; when the two circles intersect, there are only two common tangents, direct; when the two circles touch internally, the two direct common tangents become coincident; when one circle lies entirely within the other, and does not touch it, there can be no common tangent.

19. Let ABC be an equilateral triangle inscribed in a circle, and let D be any point in the arc cut off by AC:
to prove $BD = AD + CD$.

Produce CD to E, making $DE = DA$, and join AE.
Since A, B, C, D are concyclic,
$\therefore$ $\angle ADE = \angle ABC$, *III.* 22, *Cor.*
$= \frac{1}{3}$ of 2 rt. $\angle$s;
$\therefore$ the isosceles $\triangle DAE$ must be equilateral by the twelfth deduction from I. 32.

Hence in $\triangle$s BAD, CAE, $\left\{\begin{array}{rcl} BA & = & CA \\ AD & = & AE \\ \angle BAD & = & \angle CAE; \end{array}\right.$

$\therefore BD = CE$ (I. 4) $= CD + DA$.

20. Let $ACBD$ be a circle, AB, CD two chords intersecting each other perpendicularly at E.

Draw the diameter DF, and join AC, CF, FB, BD.

Because DCF is a semicircle, $\angle DCF$ is right; *III.* 31

and $\angle BEC$ is right;

$\therefore CF$ is $\parallel AB$. *I.* 28

$\therefore$ arc AC = arc BF, and chord AC = chord BF.

Now $AE^2 + EC^2 + BE^2 + ED^2 = AC^2 + BD^2$, *I.* 47

$= BF^2 + BD^2$,

$= DF^2$. *III.* 31; *I.* 47

21. Let $ABCD$ be the given quadrilateral; let the circle of which AB is a chord cut the circles of which AD and BC are chords at E, F respectively, and let the circle of which CD is a chord cut the circles of which AD and BC are chords at H, G respectively: to prove E, F, G, H concyclic.

Join AE, BF, CG, DH, and EF, FG, GH, HE.

Then $(\angle AEF + \angle ABF) + (\angle AEH + \angle ADH)$

$= 4$ rt. $\angle$s, *III.* 22

$= \angle AEF + \angle AEH + \angle FEH;$

$\therefore \angle ABF + \angle ADH = \angle FEH$.

Similarly $\angle CBF + \angle CDH = \angle FGH;$

$\therefore \angle ABF + \angle CBF + \angle ADH + \angle CDH$

$= \angle FEH + \angle FGH;$

$\therefore \angle ABC + \angle ADC = \angle FEH + \angle FGH$.

But $\angle ABC + \angle ADC = 2$ rt. $\angle$s; *III.* 22

$\therefore \angle FEH + \angle FGH = 2$ rt. $\angle$s;

$\therefore$ the points E, F, G, H are concyclic. *III.* 22

22. Let $ABCD$ be the given quadrilateral.

The centre of the circle which touches DA, AB, BC internally is found by bisecting $\angle$s A and B by AE, BE which meet at E (compare App. IV. 1). Bisect $\angle$s C and D by CG, DG which meet at G; G is the centre of the circle which touches internally BC, CD, DA. Let AE, DG meet at F, and BE, CG meet at H; F is the centre of the circle which touches CD, DA, AB internally, and H the centre of the circle which touches AB, BC, CD internally: to prove E, F, G, H concyclic.

Because $\angle AEB + \frac{1}{2}\angle A + \frac{1}{2}\angle B = 2$ rt. $\angle$s, *I.* 32
and $\angle CGD + \frac{1}{2}\angle C + \frac{1}{2}\angle D = 2$ rt. $\angle$s; *I.* 32
$\therefore \angle AEB + \angle CGD + \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$
$= 4$ rt. $\angle$s;
$\therefore \angle AEB + \angle CGD = 2$ rt. $\angle$s;
$\therefore \angle FEH + \angle FGH = 2$ rt. $\angle$s; *I.* 15
$\therefore$ the points E, F, G, H are concyclic.

The centres of the circles described outside the quadrilateral so as to touch the sides and the sides produced are found by bisecting the exterior angles at A, B, C, D, and the proof that they are concyclic is similar to the preceding.

23. Let $ABCD$ be a quadrilateral inscribed in a circle, let AB, DC meet at E, and AD, BC at F: to prove that, if EO, FO bisect $\angle$s E and F, $\angle EOF$ is right.

Let FO meet AB at G and CD at H.

Then $\angle EGO = \angle A + \frac{1}{2}\angle F$, *I.* 32
and $\angle EHO = \angle DCF + \frac{1}{2}\angle F$. *I.* 32
But $\angle DCF = \angle A$; *III.* 22, *Cor.*
$\therefore \angle EGO = \angle EHO$;
$\therefore \angle EOG = \angle EOH =$ a right angle. *I.* 32

[Other solutions will be found in Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire* (6ème éd.), p. 38, and in Mathematical Questions with their Solutions from the *Educational Times*, vol. v. p. 105.]

24. Let $ABCD$ be a quadrilateral inscribed in a circle, let AB, DC meet at E, and AD, BC at F: to prove $EF^2 =$ the sum of the squares on the tangents from E and F to the circle ABC.

Let G be the point in which the circle circumscribed about $\triangle BCE$ cuts EF; join CG.

Then $(\angle BCD + \angle BAD) + (\angle BCG + \angle BEG)$
$= 4$ rt. $\angle$s, *III.* 22
$= \angle BCD + \angle BCG + \angle DCG$;
$\therefore \angle BAD + \angle BEG = \angle DCG$.
But $\angle DFG$ is supplementary to $\angle BAD + \angle BEG$; *I.* 32
$\therefore \angle DFG$ is supplementary to $\angle DCG$;
$\therefore$ the points C, D, F, G are concyclic. *III.* 22

Now since C, D, F, G are concyclic,
$\therefore EF \cdot EG = ED \cdot EC$, *III.* 35, *Cor.*
$=$ square on tangent from E. *III.* 36

Since B, C, G, E are concyclic,

$\therefore$ $FE \cdot FG = FB \cdot FC$, *III.* 35, *Cor.*

$= \text{square on tangent from } F$; *III.* 36

and $EF \cdot EG + FE \cdot FG = EF^2$. *II.* 2

25. Let ABC be a triangle (for convenience let AB be greater than AC), and let PQ be the diameter of the circumscribed circle which is $\perp$ BC, Q being on the same side of BC as A; from P, Q let there be drawn PS, PT, and QU, QV respectively $\perp$ AB, AC:

to prove $AS = AT = BU = CV = \frac{1}{2}(AB + AC)$,

and $AU = AV = BS = CT = \frac{1}{2}(AB - AC)$.

Join P, Q with the vertices A, B, C.

Since PQ is a diameter and is $\perp$ BC,

$\therefore$ the arcs BP, CP are equal, and the chords BP, CP, as also the chords BQ, CQ.

Because arc BP = arc CP, $\therefore$ $\angle\, SAP = \angle\, TAP$; *III.* 27

$\therefore$ the right-angled $\triangle$s ASP, ATP are congruent, *I.* 26

and $AS = AT$, $SP = TP$.

Since $BP = CP$, $SP = TP$, and $\angle$s at S and T are right,

$\therefore$ $BS = CT$. *I.* 47

Again since AP bisects $\angle\, BAC$, and AQ is $\perp$ AP; *III.* 31

$\therefore$ AQ bisects $\angle\, UAV$;

$\therefore$ the right-angled $\triangle$s AUQ, AVQ are congruent, *I.* 26

and $AU = AV$, $QU = QV$.

Since $BQ = CQ$, $QU = QV$, and $\angle$s at U and V are right,

$\therefore$ $BU = CV$. *I.* 47

$$\begin{aligned} \text{Now } \tfrac{1}{2}(AB + AC) &= \tfrac{1}{2}\left\{(AS + BS) + (AT - CT)\right\}, \\ &= \tfrac{1}{2}\left\{(AS + AT) + (BS - CT)\right\}, \\ &= \tfrac{1}{2}(2AS) = \tfrac{1}{2}(2AT), \\ &= AS = AT. \end{aligned}$$

$$\begin{aligned} \text{Also } \tfrac{1}{2}(AB + AC) &= \tfrac{1}{2}\left\{(BU + AU) + (CV - AV)\right\}, \\ &= \tfrac{1}{2}\left\{(BU + CV) + (AU - AV)\right\}, \\ &= \tfrac{1}{2}(2BU) = \tfrac{1}{2}(2CV), \\ &= BU = CV. \end{aligned}$$

Similarly $\frac{1}{2}(AB - AC) = AU = AV = BS = CT$.

26. To prove $\frac{1}{2}(\angle\, C + \angle\, B) = QBC = QCB = QAB$
$= QPB = QPC$,

$\frac{1}{2}(\angle\, C - \angle\, B) = ABQ = ACQ = APQ$
$= BPS = CPT$.

Because $\angle BQC = \angle BAC$, *III.* 21

$\therefore \quad \angle C + \angle B = \angle QBC + \angle QCB$; *I.* 32

$\therefore \frac{1}{2}(\angle C + \angle B) = \angle QBC = \angle QCB$,

$= \angle QPC = \angle QPB = \angle QAB$. *III.* 21

Since half the sum of two magnitudes diminished by half their difference gives the less, and $\angle QBC$ diminished by $\angle QBA$ gives $\angle B$,

$\therefore \frac{1}{2}(\angle C - \angle B) = \angle QBA$,

$= \angle QCA = \angle QPA$. *III.* 21

Because $\angle QBP$ is right, *III.* 31

$\therefore \quad \angle QBA$ is complementary to $\angle SBP$.

But $\angle BPS$ is complementary to $\angle SBP$;

$\therefore \angle QBA = \angle BPS = \angle CPT$.

27. From Q, any point in the $\bigcirc^{ce}$ of the circle circumscribed about $\triangle ABC$, let QX, QY, QZ be drawn $\perp$ BC, CA, AB respectively: to prove X, Y, Z collinear.

Join XZ, YZ, AQ, BQ.

Since $\angle$s QZB, QXB are right, Q, Z, X, B are concyclic;

$\therefore \quad \angle BQX = \angle BZX$. *III.* 21

Since $\angle$s QYA, QZA are right, Q, Y, A, Z are concyclic;

$\therefore \quad \angle AQY = \angle AZY$. *III.* 21

Since $\angle$s QYC, QXC are right, Q, Y, C, X are concyclic;

$\therefore \quad \angle XQY =$ supplement of $\angle C$. *III.* 22

But $\angle AQB =$ supplement of $\angle C$; *III.* 22

$\therefore \quad \angle AQB = \angle XQY$;

$\therefore \quad \angle BQX = \angle AQY$;

$\therefore \quad \angle BZX = \angle AZY$;

$\therefore$ X, Z, Y are collinear, by the second deduction from I. 14.

28. From Q, any point in the $\bigcirc^{ce}$ of the circle circumscribed about $\triangle ABC$, let QX, QY, QZ be drawn making with BC, CA, AB the equal angles QXC, QYC' (C' is on CA produced), QZA: to prove X, Y, Z collinear.

Join XZ, YZ, AQ, BQ.

Since $\angle$s QZB, QXB are equal, being supplementary to the equal $\angle$s QZA, QXC, $\therefore$ Q, Z, X, B are concyclic;

$\therefore \quad \angle BQX = \angle BZX$. *III.* 21

Since $\angle QYA$ is supplementary to $\angle QYC'$, it is also supplementary to $\angle QZA$; $\therefore$ Q, Y, A, Z are concyclic;

$\therefore \quad \angle AQY = \angle AZY$. *III.* 21

Since $\angle QYC$ is supplementary to $\angle QYC'$, it is also supplementary to $\angle QXC$; $\therefore$ Q, Y, C, X are concyclic;

$\therefore \quad \angle XQY =$ supplement of $\angle C$. *III.* 22

But $\angle AQB =$ supplement of $\angle C$; *III.* 22

$\therefore \quad \angle AQB = \angle XQY$;

$\therefore \quad \angle BQX = \angle AQY$;

$\therefore \quad \angle BZX = \angle AZY$;

$\therefore$ X, Y, Z are collinear, by the second deduction from I. 14.

29. Join PA, PB, PC.

Because $\angle$s AZP, AYP are right, the points A, Z, P, Y are concyclic, and the circle which contains them has O_1, the middle point of PA, for centre, and is the circle circumscribed about $\triangle AZY$.

Because $\angle$s BXP, BZP are right, the points B, X, Z, P are concyclic, and the circle which contains them has O_2, the middle point of PB, for centre, and is the circle circumscribed about $\triangle BXZ$.

Because $\angle$s CYP, CXP are right, the points C, Y, P, X are concyclic, and the circle which contains them has O_3, the middle point of PC, for centre, and is the circle circumscribed about $\triangle CYX$.

In $\triangle PAB$, O_1O_2 is $\parallel AB$ and $= \frac{1}{2} AB$;

in $\triangle PBC$, O_2O_3 is $\parallel BC$ and $= \frac{1}{2} BC$;

in $\triangle PCA$, O_3O_1 is $\parallel CA$ and $= \frac{1}{2} CA$; *App. I.* 1

$\therefore$ $\triangle O_1O_2O_3$ has its sides respectively the halves of the sides of $\triangle ABC$.

But the sides of $\triangle ABC$ are fixed lengths;

$\therefore$ the sides of $\triangle O_1O_2O_3$ are fixed lengths, and the $\triangle O_1O_2O_3$ is constant in magnitude.

Hence also the circle circumscribed about $\triangle O_1O_2O_3$ is constant in magnitude.

30. Let ABC be a triangle whose sides BC, CA, AB are cut by a transversal in X, Y, Z respectively; about $\triangle$s AYZ, BXZ let circles be circumscribed intersecting in Q:
to prove Q, Y, C, X concyclic, and also Q, A, C, B.

Join Q to X, Y, Z, A, B.

Because $\angle XQZ = \angle XBZ$, *III.* 21

and $\angle ZQY =$ supplement of $\angle ZAY$, *III.* 22

$= \angle ZAC$;

$\therefore \quad \angle XQZ + \angle ZQY = \angle XBZ + \angle ZAC$;

$\therefore \quad \angle XQY =$ supplement of $\angle C$; *I.* 32

$\therefore$ the points Q, Y, C, X are concyclic. *III.* 22

Again $\angle BQX = \angle BZX$, *III.* 21

$= \angle AZY$, *I.* 15

$= \angle AQY$; *III.* 21

$\therefore \angle BQX + \angle AQX = \angle AQY + \angle AQX$;

$\therefore \angle AQB = \angle XQY$,

$=$ supplement of $\angle C$; *III.* 22

$\therefore$ the points Q, A, C, B are concyclic. *III.* 22

31. Let A be the given circle, P and Q the two given points. Suppose two circles B and C to pass through P and Q, and to intersect the circle A.

Since B and C pass through P and Q,

$\therefore PQ$ produced is the radical axis of B and C.

The straight line joining the intersections of A and B is the radical axis of A and B; let it meet PQ or PQ produced at O. Then O is the radical centre of A, B, and C, and therefore is a fixed point.

32. Let AB be the fixed straight line, P the fixed point in it, and let a circle whose centre is O touch AB at P, and be cut at E by the parallel fixed straight line CD: to prove that EF, the tangent at E, touches a fixed circle.

From P draw $PQ \perp CD$, and passing through O. *III.* 19

Join OE, PE, and draw $PF \perp EF$.

Then $\angle OPE = \angle OEP$, *I.* 5

$= \angle EPF$; *I.* 29

$\therefore$ the right-angled $\triangle$s PQE, PFE are congruent, *I.* 26

and $PQ = PF$.

Similarly $PQ =$ the perpendiculars from P on the tangents to the other circles at the points where they are cut by CD. Hence all these tangents touch the circle described with the fixed centre P, and the fixed radius PQ.

33. Produce DA to E, making $AE = AD$, and join EB.

Then $\angle AEB = \angle ABE$; *I.* 5

$\therefore \angle BCD + \angle AEB = \angle CDA + \angle CBA + \angle ABE$,

$= \angle CDE + \angle CBE$.

Now the four angles of $BCDE = 4$ rt. $\angle$s; *I.* 32, *Cor.* 2

$\therefore \angle BCD + \angle AEB = 2$ rt. $\angle$s;

$\therefore$ the points B, C, D, E lie on the $\bigcirc^{ce}$ of a circle. *III.* 22

Now since $AB = AD = AE$,

$\therefore A$ is the centre of this circle; *III.* 9

$\therefore AC = AB$ or AD.

In this solution the quadrilateral $ABCD$ is supposed to be

convex. (See Note 2 on p. 249 of *Euclid.*) But the theorem is true when the quadrilateral is re-entrant. The proof of it varies little from the preceding, the principal change being that $\angle$s BCD and AEB, instead of being shown to be supplementary, are shown to be equal.

34. Let CF produced meet AB at G.

Then $\angle CDF = \angle DAB$, *III.* 32

and $\angle CEF = \angle EBA$; *III.* 32

$\therefore \angle CDF + \angle CEF = \angle DAB + \angle EBA$,

$= \angle DAB + \angle ABD + \angle FBE$.

Now since $\angle$s ADB, AEB are right, *III.* 31

$\therefore \angle DAB + \angle ABD = \angle FEB$;

$\therefore \angle CDF + \angle CEF = \angle FEB + \angle FBE$,

$= \angle DFE$. *I.* 32

Hence $CF = CD$, by the preceding deduction;

and $\angle CFD = \angle CDF$, *I.* 5

$= \angle DAG$. *III.* 32

But $\angle CFD + \angle DFG = 2$ rt. $\angle$s; *I.* 13

$\therefore \angle DAG + \angle DFG = 2$ rt. $\angle$s;

$\therefore \angle ADF + \angle AGF = 2$ rt. $\angle$s; *I.* 32, *Cor.* 2

$\therefore \angle AGF$ is right.

35. Let $F'C$ produced meet AB at G. Join AE, BD.

Then $\angle$s ADB, AEB are right; *III.* 31

$\therefore \angle CDF' =$ complement of $\angle CDB$,

$=$ complement of $\angle DAB$. *III.* 32

Similarly $\angle CEF' =$ complement of $\angle EBA$;

$\therefore \angle CDF' + \angle CEF' = 2$ rt. $\angle$s $- (\angle DAB + \angle EBA)$,

$= \angle DF'E$; *I.* 32

$\therefore CF' = CD$, by the thirty-third deduction;

$\therefore \angle CF'D = \angle CDF'$. *I.* 5

Now $\angle CDF' =$ complement of $\angle DAB$,

$= \angle DBA$;

$\therefore$ the points D, F', B, G are concyclic; *III.* 21

$\therefore \angle F'GB = \angle F'DB$, *III.* 21

$=$ a right angle.

36. If the tangents to a series of circles from a fixed point are equal, the point has equal potencies with respect to the circles;

$\therefore$ it must be situated on the radical axes of every pair of the circles, that is, on their common chords produced.

37. Let AB, AC be the two straight lines at right angles to each other.

Draw any straight line forming with AB, AC an isosceles triangle, by the third deduction from I. 31; parallel to that straight line draw a tangent to the given circle, by the sixth deduction from III. 16. Let GDH be the tangent, G and H being on AB and AC, and D being the point of contact.

First let the circle be outside $\triangle\ GAH$. Draw DE, DF $\perp AB$, AC respectively. Take any other point D' on the $\bigcirc^{ce}$ and draw $D'E'$, $D'F' \perp AB$, AC respectively, and through D' draw $G'H' \parallel GH$.

Since $\triangle\ AGH$ is isosceles, and DF is $\parallel GA$,

$\therefore\ \triangle\ FDH$ is isosceles, and $DF = HF$;

$$\therefore \quad DE + DF = FA + HF, \qquad I.\ 34$$
$$= AH.$$

Similarly $D'E' + D'F' = AH'$.

But AH is less than AH';

$\therefore\ DE + DF$ is less than $D'E' + D'F'$.

Second, let the circle be inside $\triangle\ GAH$. Make the same construction as before, and a similar method of proof will show that $DE + DF$ is greater than $D'E' + D'F'$.

38. Let A be the given point on the $\bigcirc^{ce}$, BC the given chord.

Find O the centre of the given circle, join AO, and on AO as diameter describe a circle cutting BC at D. Join AD, and produce it to meet the $\bigcirc^{ce}$ at E.

AE is the required chord.

Join OD.

Then $\angle\ ODA$ is right, *III.* 31

$\therefore\ AE$ is bisected at D. *III.* 3

39. Find E the centre of the circle, join EO, OP, PE, and from O draw $OF \perp PE$. Through the points A, O, B describe a circle, and let it meet PO again at G, and join BG.

Then $\angle\ BGO = \angle\ PAO$, *III.* 22, *Cor.*

$= \angle\ PDB$, by the second deduction from III. 21;

$\therefore$ the points P, D, G, B are concyclic; *III.* 21

$\therefore\quad PO \cdot OG = BO \cdot OD$. *III.* 35

$$\begin{aligned} \text{Now } PA \cdot PB &= PO \cdot PG, && III.\ 35, Cor. \\ &= PO^2 + PO \cdot OG, && II.\ 3 \\ &= PO^2 + BO \cdot OD, \\ &= PE^2 + OE^2 - 2\,PE \cdot EF + BO \cdot OD. && II.\ 13 \end{aligned}$$

If EA, EB be joined, $\triangle\ EAB$ is isosceles,

and $PE^2 = EA^2 + PA \cdot PB$, by the first deduction of Book II.;

if ED also be joined, $\triangle$ EBD is isosceles,
and $OE^2 = EB^2 - BO \cdot OD$, by the first deduction of Book II.
Hence $PA \cdot PB = EA^2 + PA \cdot PB + EB^2 - BO \cdot OD$
$$- 2\,PE \cdot EF + BO \cdot OD,$$
$$= PA \cdot PB + 2AE^2 - 2\,PE \cdot EF;$$
$\therefore$ $AE^2 = PE \cdot EF$,
a result which proves (see the eighth deduction from III. 17, and the second part of the fourteenth deduction of Book II.) that O lies on the chord of contact of the tangents from P.
[This solution is due to Hugh Hamilton Browning.]

40. Let P be a point outside a given circle.

From P draw three secants PAB, PCD, PEF; join AD, CB intersecting in G; join CF, ED intersecting in H; and let GH produced meet the $\bigcirc^{ce}$ at K and L.

PK, PL are the required tangents.

By the preceding deduction, G is on the chord of contact of the tangents from P, and so is H;
$\therefore$ GH produced is the chord of contact;
$\therefore$ PK, PL are the tangents from P.

LOCI.

1. The locus is the perpendicular to the given straight line at the given point, produced indefinitely either way. *III.* 16
2. The locus is the straight line joining the centre of the given circle to the given point, produced indefinitely either way. *III.* 11, 12
3. The locus is the same as the first locus of Book I.
4. The locus is the same as the second locus of Book I.
5. The locus is the same as the third or fourth locus of Book I.
6. Let A and B be the centres of the two equal circles.

Join AB; bisect AB at C; and draw through C a perpendicular to AB. This perpendicular is the locus.

Take any point D in the perpendicular; join DA, DB cutting the circles at E and F.

Then $AD = BD$, by the sixth deduction from I. 4,
and $AE = BF$;
$\therefore$ $DE = DF$.
Hence a circle with centre D and radius DE will touch both circles.

7. The middle points of a series of parallel chords lie on the diameter perpendicular to the chords. *III.* 3
8. See the first deduction from III. 14.
9. The locus is the $\bigcirc^{ce}$ of the circle described on the hypotenuse as diameter, by the tenth deduction from III. 31.
10. Let A be the given point, BDF the given circle.

    Through A draw any chord BD; bisect BD at E, join E with the centre C, and join AC.

    Then $\angle\ CEA$ is right; *III.* 3

    $\therefore$ the locus is the $\bigcirc^{ce}$ of the circle described on AC as diameter, by the preceding deduction.

    If the given fixed point be situated inside the given circle, or on its $\bigcirc^{ce}$, the locus is the complete $\bigcirc^{ce}$; if the fixed point be situated outside the given circle, the locus consists only of that arc of the $\bigcirc^{ce}$ which is included within the given circle.
11. The locus consists of the arcs of two segments each containing an angle equal to the given angle, and described on opposite sides of the given base.
12. The locus consists of the arcs of two segments each containing an angle equal to $\frac{1}{2}\ \angle\ A$, and described on opposite sides of BC. See the first part of the sixteenth deduction of Book III.
13. The locus consists of the arcs of two segments, each containing an angle equal to a right angle $+\ \frac{1}{2}\ \angle\ A$, and described on opposite sides of BC.

    See the second part of the sixteenth deduction of Book III.
14. Suppose arc ACB to be less than a semicircle.

    If C be joined to the points of contact of the tangents PA, PB, and also to A and B, it may be proved that CA and CB bisect $\angle$s PAB, PBA.

    Now $\angle\ P = 2$ rt. $\angle$s $-\ 2\,(\angle\ CAB + \angle\ CBA)$, *I.* 32

    $= 2$ rt. $\angle$s $-\ 2\,(2$ rt. $\angle$s $-\ \angle\ ACB)$, *I.* 32

    $= 2\,\angle\ ACB - 2$ rt. $\angle$s.

    But $\angle\ ACB$ is a fixed angle, since AB is a fixed chord;

    $\therefore\ \angle\ P$ is a fixed angle, and the locus of P is the arc of a segment of a circle described on AB as base.

    When arc ACB is greater than a semicircle, it will be found that CA and CB do not bisect $\angle$s PAB, PBA, but the angles adjacent to them, and that the locus of P is still the arc of a segment on AB as base. When arc ACB is a

semicircle, the point P is infinitely distant, for the tangents are then parallel.

15. Let $ABCD$ be the quadrilateral, having AB fixed and CD a constant length, and let AD, BC meet at O.

Join AC.

Then $\angle AOB = \angle ACB - \angle DAC$. *I.* 32

Now $\angle ACB$ is constant, since AB is fixed,

and $\angle DAC$ is constant, since CD is constant;

$\therefore$ $\angle AOB$ is constant, and the locus of O is an arc of a segment of a circle on AB as base.

Again, let AC, BD meet at I.

Then $\angle AIB = \angle ACB + \angle DBC$; *I.* 32

$\therefore$ $\angle AIB$ is constant, and the locus of I is an arc of a segment of a circle on AB as base.

16. Let GH be the given straight line, D and E the given points in it, and let A and B be two circles touching GH at D and E, and each other at C: to find the locus of C.

At C draw CF a common tangent to the two circles, and let CF meet DE at F.

Then $FD = FC = FE$; *III.* 17, *Cor.*

$\therefore$ the common tangent at C bisects the fixed distance DE, and is equal to half DE.

Hence the locus of C is the arc of a semicircle on DE as diameter.

17. Let BDC (fig. to III. 17) be a circle, and suppose A to be a point such that the tangents AB, AC contain a right angle.

Join AE, BE, CE.

Then $\angle BAE$ = half a right angle, by the second deduction from III. 17.

$\therefore$ $\angle AEB$, which is the complement of $\angle BAE$, is known; and the length of the radius EB is known;

$\therefore$ $\triangle ABE$ could be constructed from what is known, by the ninth deduction from I. 23;

in other words, EA is a constant length.

Now since E is a fixed point, and EA a constant length, the locus of A is the $\bigcirc^{ce}$ of a circle concentric with BDC.

18. Let BDC (fig. to III. 17) be a circle, and suppose A to be a point such that the tangents AB, AC contain a given angle.

Join AE, BE, CE.

Then $\angle BAE$ = half of the given angle, by the second deduction from III. 17;

$\therefore \angle AEB$, which is the complement of $\angle BAE$, is known; and the length of the radius EB is known;

$\therefore \triangle ABE$ could be constructed from what is known, by the ninth deduction from I. 23;

in other words EA is a constant length.

Now since E is a fixed point, and EA a constant length, the locus of A is the $\bigcirc^{ce}$ of a circle, concentric with BDC.

19. Let BDC (fig. to III. 17) be a circle, and suppose A to be a point such that the tangents AB, AC are of given length.

Join AE, BE.

Since the lengths of AB and EB are known,

$\therefore$ the right-angled $\triangle ABE$ could be constructed from what is known, by the eleventh deduction from I. 23;

in other words, EA is a constant length.

Now since E is a fixed point, and EA a constant length, the locus of A is the $\bigcirc^{ce}$ of a circle concentric with BDC.

20. The statement of this deduction should be:

A is the given point on the $\bigcirc^{ce}$ of a given circle, and AQ is any chord drawn through it. Find the locus of a point P taken on AQ or AQ produced such that $AP \cdot AQ$ is constant.

Through A draw the diameter AE; AE is a fixed straight line.

From P draw $PD \perp AE$ or AE produced; and join QE.

Because $\angle$s PQE, PDE are right, *III.* 31

$\therefore$ the points P, D, E, Q are concyclic; *III.* 22

$\therefore AD \cdot AE = AP \cdot AQ,$ *III.* 35, *Cor.*

$= $ a fixed magnitude;

$\therefore AD$ is a fixed length, and D a fixed point.

Hence every straight line drawn from the fixed point D to any point P on the locus is $\perp AD$;

$\therefore$ the locus of P is the perpendicular drawn to AD through D.

21. From A draw $AD \perp BC$; AD is a fixed straight line.

From Q draw $QE \perp AP$, and meeting AD at E.

Because $\angle$s PQE, PDE are right,

$\therefore$ the points P, D, E, Q are concyclic; *III.* 22

$\therefore AD \cdot AE = AP \cdot AQ,$ *III.* 35, *Cor.*

$= $ a fixed magnitude;

$\therefore AE$ is a fixed length, and E a fixed point.

Hence every straight line drawn from the fixed point E to any point Q on the locus is $\perp AQ$;

$\therefore$ the locus of Q is the $\bigcirc^{ce}$ of the circle described on AE as diameter.

22. Let ABC (fig. to I. 47) be one of the right-angled triangles whose hypotenuse BC is given: to find the loci of G and H, F and K.

Since $\angle$s BGC, BHC are each half a right angle, the locus of G and H is the arc of a segment of a circle described on BC, and containing an angle equal to half a right angle.

Join AF, AK, which are in the same straight line, by the twenty-second deduction from I. 47; from F and K draw FM, $KN \perp BC$ produced; bisect BC at A', through A' draw $A'L' \perp BC$, meeting FK at L', and join BL', CL'.

Because N and M are equidistant in opposite directions from B and C, by the thirty-fourth deduction from I. 47;
$\therefore$ they are equidistant from A';
$\therefore$ the middle point of MN is a fixed point.
Because MF, $A'L'$, NK are parallel, and A' is the middle point of MN,
$\therefore$ L' is the middle point of FK,
and $A'L' = \frac{1}{2}(FM + KN)$, *App. I.* 1, *Cor.* 2
$= \frac{1}{2} BC$, by the thirty-fourth deduction from I. 47,
that is, $=$ a fixed length;
$\therefore$ L', the middle point of FK, is a fixed point, and BL', CL' are fixed straight lines.
Now since $\angle$s BFL', CKL' are each half a right angle, the loci of F and K are two arcs of segments of circles described on BL', CL', and containing an angle equal to half a right angle.

23. Let P be any point inside a given circle, whose centre is O, and let AB be any chord passing through P; let AC, BC, the tangents at A and B, meet at C: to find the locus of C.

Join OP, and from C draw $CQ \perp OP$ produced; join OA, OC, and let OC meet AB at D.

Then $\angle\ OAC$ is right; *III.* 18
AB is $\perp OC$, by the eighth deduction from III. 17;
and $OC \cdot OD = OA^2$, by the second part of the fourteenth deduction of Book II.
Now $\angle$s PDC, PQC are right;
$\therefore$ the points C, D, P, Q are concyclic; *III.* 22

$\therefore\ OP \cdot OQ = OD \cdot OC,$ III. 35, Cor.

$= OA^2 =$ a fixed magnitude.

But OP is a fixed length;

$\therefore\ OQ$ is also a fixed length, and Q a fixed point.

Hence every straight line drawn from the fixed point Q to any point C on the locus is $\perp OQ$;

$\therefore\ QC$ produced indefinitely is the locus.

24. When the point P is outside the circle, the preceding construction and proof are applicable word for word, the only changes necessary being the omission of 'produced' from the construction, and the substitution of III. 21 for III. 22 in the proof.

BOOK IV.

PROPOSITION 1.

1. There can be drawn as many chords as there are diameters of the circle, that is, an infinite number.
2. Let C be the given point in the $\bigcirc^{ce}$.
   Draw the diameter CB, and complete the construction as in the proposition.
   Two chords CA and CF.
3. Let AB be the given chord.
   Bisect AB perpendicularly by CD.
   Then any point in CD will be the centre of a circle which can be circumscribed about AB; because every point in CD is equidistant from A and B.
   Hence there can be an infinite number of such circles.
   The diameters of these circles may not be less than AB, but they may be as much larger than AB as we please.
4. Let AB be the given chord, M the given radius.
   Bisect AB perpendicularly by CD; with A or B as centre and M as radius cut CD at the two points E, F.
   E or F is the centre of the required circle, EA, FA, or EB, FB is the required radius.
   Only two circles.

5, 6. See the fourth deduction from III. 18.

7. Place a chord AC in the given circle $ABC = D$.

From O, the centre of circle ABC, draw $OE \perp AC$; with O as centre and OE as radius describe a circle, and to this circle draw a tangent $\parallel$ the given straight line, by the sixth deduction from III. 16.

The proof follows from the second deduction from III. 18.

8. Repeat the previous construction, substituting $\perp$ for $\parallel$, and 'seventh' for 'sixth.'

PROPOSITION 2.

1. Because there are innumerable points such as A, that may be taken on the $\bigcirc^{ce}$ of the given circle.
2. For $\angle HAC$ may be made $= \angle D$, and then $\angle GAB =$ either $\angle E$ or $\angle F$; or $\angle HAC$ may be made $= \angle E$, and then $\angle GAB =$ either $\angle F$ or $\angle D$; or $\angle HAC$ may be made $= \angle F$, and then $\angle GAB =$ either $\angle D$ or $\angle E$.
3. Take any straight line EF, and on it describe an equilateral $\triangle DEF$. In the circle ABC inscribe a triangle equiangular to $\triangle DEF$.
4. Since $\angle A = \angle L$, $\therefore BC = MN$. *III.* 26, 29

   Similarly $CA = NL$, and $AB = LM$;

   $\therefore$ $\triangle$s ABC, LMN are congruent. *I.* 8

PROPOSITION 3.

1. If the tangents at A and B do not meet, they are parallel; $\therefore$ AO and BO, which are $\perp$ the tangents, are also parallel, which is absurd.

   Hence also the tangents at B and C will meet, and the tangents at C and A.
2. Because there are innumerable radii such as OB that may be drawn in the given circle.
3. Take any straight line EF, and on it describe an equilateral $\triangle DEF$. About the circle ABC circumscribe a triangle equiangular to $\triangle DEF$.

4, 5. Let LMN (fig. to IV. 3) be a circumscribed equilateral triangle, and let AB, BC, CA be joined.

Then $\angle LAC + \angle LCA = 2$ rt. $\angle$s $- \angle L$
$= \angle M + \angle N$. *I.* 32

Now $\angle LAC = \angle LCA$ (III. 17, Cor., I. 5) and $\angle M = \angle N$;
$\therefore$ $\angle LAC = \angle M$, $\angle LCA = \angle N$, and AC is $\parallel MN$.
Hence also AB is $\parallel LN$, and $BC \parallel ML$;
$\therefore$ $\triangle ABC$ has its sides respectively $\parallel$ the sides of $\triangle LNM$;
$\therefore$ $\triangle ABC$ is equiangular to $\triangle LNM$, and consequently equilateral.

Since $LABC$, $MBCA$, $NCAB$ are $\parallel^{\text{ms}}$,
$\therefore$ $AL = BC = AM$, $BM = AC = BN$, $CN = AB = CL$.
Also $\triangle ABC =$ each of the $\triangle$s LAC, MBA, NCB;
$\therefore$ $\triangle LMN =$ four times $\triangle ABC$.

6. Since $\angle$s OAM, OBM are right,
$\therefore$ the quadrilateral $AOBM$ is inscriptible in a circle;
$\therefore$ $\angle GOA = \angle M$; *III.* 22, *Cor.*
$\therefore$ $\angle M = \angle E$.
Hence also $\angle N = \angle F$; and $\therefore$ $\angle L = \angle D$.

7. For RP, LM are both $\perp OA$, and therefore parallel;
$\therefore$ $\angle LMN = \angle RPQ = \angle E$.
Hence also $\angle LNM = \angle F$; $\therefore$ $\angle MLN = \angle D$.

8. Let A', B', C' be the points of bisection respectively of the arcs BC, CA, AB, and let LMN be the triangle formed by drawing tangents at A', B', C'.

From O, the centre of the circle, draw a perpendicular to BC; this perpendicular will bisect BC (III. 3), and also the arc BC (III. 30), and will therefore pass through A';
$\therefore$ MN and BC are both $\perp OA'$; $\therefore$ MN is $\parallel BC$.
Hence also NL is $\parallel CA$, and $LM \parallel AB$;
$\therefore$ $\triangle LMN$ is equiangular to $\triangle ABC$, and consequently to $\triangle DEF$.

9. Let $LMNPQ$ be the resulting figure.
Then it may be proved, as in the preceding deduction, that LM is $\parallel AB$, $MN \parallel BC$, $NP \parallel CD$, $PQ \parallel DE$, and $QL \parallel EA$; and these pairs of parallels are drawn in similar directions;
$\therefore$ the angles of $LMNPQ$ are respectively equal to those of $ABCDE$.

10. Let $L'M'N'$ be the other triangle, A', B', C' the points of contact of its sides with the circle; then if $\angle A'OB' = \angle DEG$, $B'OC' = DFH$, it will follow that $\angle A'OB' = \angle AOB$, $\angle B'OC' = \angle BOC$, $\angle C'OA' = \angle COA$.
Now if OA', OB', OC', while retaining their relative positions,

be rotated round O till they coincide with OA, OB, OC, $\triangle\ L'M'N'$ will then coincide with $\triangle\ LMN$.

11. Let ABC be the given circle, and DEF the given triangle. Find O the centre of the circle, and draw any radius OB. Make $\angle\ BOA = \angle\ DEF$, and $\angle\ BOC = \angle\ DFE$; at A, B, C draw tangents to the circle intersecting each other at L, M, N. LMN is the required triangle.

Since $\angle$s OAM, OBM are right, *III.* 18
$\therefore$ the points O, A, M, B are concyclic; *III.* 22
$\therefore\ \angle\ LMN = \angle\ AOB = \angle\ DEF$. *III.* 22, *Cor.*
Similarly $\angle\ LNM = \angle\ DFE$;
$\therefore$ remaining $\angle\ L =$ remaining $\angle\ D$.

The other two solutions are obtained thus:

Let ABC be the given circle, and DEF the given triangle.

Produce FE to G. Find O the centre of the circle, and draw any radius OB.
Make $\angle\ BOA = \angle\ DEG$, and $\angle\ BOC = \angle\ DFE$;
at A, B, C draw tangents to the circle intersecting each other at L, M, N. LMN is the required triangle.

Let ABC be the given circle, and DEF the given triangle.

Produce EF to H. Find O the centre of the circle, and draw any radius OB.
Make $\angle\ BOA = \angle\ DEF$, and $\angle\ BOC = \angle\ DFH$;
at A, B, C draw tangents to the circle intersecting each other at L, M, N. LMN is the required triangle.

The possibility of three solutions to this deduction may also be seen from the fact that the given circle may be included either between the sides of the angle L, the sides of the angle M, or the sides of the angle N, of the constructed triangle LMN.

PROPOSITION 4.

1. Because $\angle\ ABC + \angle\ ACB$ is less than 2 rt. $\angle$s, *I.* 17
$\therefore\ \angle\ IBC + \angle\ ICB$ is less than 2 rt. $\angle$s;
$\therefore$ BI and CI must meet. *I.* 29, *Cor.*
2. For $\triangle$s EAI, FAI are congruent; *I.* 8
$\therefore\ \angle\ EAI = \angle\ FAI$.
3. Let $\triangle\ ABC$ (fig. to IV. 4) be equilateral.
Because $\angle\ ABC = \angle\ ACB$, $\therefore\ \angle\ IBC = \angle\ ICB$; $\therefore IB = IC$.
Hence also $IC = IA$; $\therefore IA = IB = IC$.

4. Let $\triangle\ ABC$ (fig. to IV. 4) be isosceles, and let $AB = AC$.
Because $\angle ABC = \angle ACB$; $\therefore \angle IBC = \angle ICB$; $\therefore IB = IC$.

5. $(AF + FB) + (BD + DC) + (CE + EA) =$ the perimeter of $\triangle\ ABC$.
But $AF = EA$, $BD = FB$, $CE = DC$; *III.* 17, *Cor.*
$\therefore AF + BD + CE = FB + DC + EA =$ the semiperimeter.

6. $AF + BD + CE =$ the semiperimeter, by the previous deduction, and $CE = CD$;
$\therefore AF + BD + CD =$ the semiperimeter;
$\therefore AF + BC =$ the semiperimeter.
Hence also $BD + CA = CE + AB =$ the semiperimeter.

7. The radii of the respective circles are AF, BD, CE.

8. The centre of the circle inscribed in the triangle will be the required centre, because it is equidistant from the three sides.

9. Let the parallel through I meet AB, AC in G, H.
Then $\angle GIB = \angle IBD = \angle IBG$; $\therefore GI = GB$.
Hence also $HI = HC$; $\therefore GH = GB + HC$.
If through I_1 there be drawn $G_1H_1 \parallel BC$ to meet AB, AC produced at G_1, H_1; then $\angle G_1I_1B = \angle I_1BD_1 = I_1BG_1$; $\therefore G_1I_1 = G_1B$.
Hence also $H_1I_1 = H_1C$; $\therefore G_1H_1 = G_1B + H_1C$.
If through I_2 there be drawn $G_2H_2 \parallel BC$ to meet AB, AC at G_2, H_2; then $\angle G_2I_2B = \angle I_2BD_2 = \angle I_2BG_2$; $\therefore G_2I_2 = G_2B$.
Hence also $H_2I_2 = H_2C$; $\therefore G_2H_2 = G_2B - H_2C$.
If through I_3 there be drawn $G_3H_3 \parallel BC$ to meet AB, AC produced at G_3, H_3; then $\angle G_3I_3B = \angle I_3BD_3 = \angle I_3BG_3$; $\therefore G_3I_3 = G_3B$.
Hence also $H_3I_3 = H_3C$; $\therefore G_3H_3 = G_3B - H_3C$.

10. For $\angle AFE = \angle FDE$ (*III.* 32), and $\angle AFE$, being one of the angles at the base of the isosceles $\triangle\ AFE$, is acute, by the fifth deduction from I. 17.
Hence also $\angle$s DEF, EFD are acute.

11. For twice $\angle FDE = \angle AEF + \angle AFE$,
$= 2$ rt. $\angle$s $- \angle A$;
$\therefore \quad \angle FDE = 1$ rt. $\angle - \frac{1}{2} \angle A$.

12. The vertex A and the centres of the circles inscribed in $\triangle$s ABC, ADE lie on the bisector of $\angle BAC$.

13. Let BD, CE be the two straight lines, and suppose A the point at which they would meet if they could be produced.

Then A and the centres of the circles inscribed in $\triangle$s ABC, ADE are collinear.
The centre of the circle inscribed in $\triangle$ ABC is found by bisecting $\angle$s ABC and ACB; the centre of the circle inscribed in $\triangle$ ADE is found by bisecting $\angle$s ADE and AED. The straight line joining these two centres would, if produced, bisect $\angle$ BAC.

PROPOSITION 5.

1. If LS, KS do not meet, they are parallel;
   $\therefore$ AB and AC, which are $\perp$ LS and KS, are also parallel, which is absurd.
2. III. 25.
3. Bisect the hypotenuse. The middle point of it is the centre, and the half of the hypotenuse is the radius, of the circumscribed circle.
4. The isosceles triangle must have its vertical angle right; its base, therefore, will be the hypotenuse of a right-angled triangle, and the diameter of the circle circumscribed about the triangle.
5. Divide the quadrilateral into two triangles, and circumscribe a circle about one of these triangles. This circle will also be circumscribed about the quadrilateral. *III.* 22
6. In $\triangle$s SBH, SCH, $\left\{\begin{array}{l}\angle\ SBH = \angle\ SCH \\ \angle\ SHB = \angle\ SHC \\ \quad SH = SH;\end{array}\right.$ *I.* 5
   $\therefore$ $BH = CH$. *I.* 26
7. (*a*) Since SH bisects BC perpendicularly,
   $\therefore$ it bisects arc BDC; $\therefore$ $\angle\ BSD = \angle\ CSD$.
   Now $\angle\ BSC =$ twice $\angle\ BAC$; *III.* 20
   $\therefore$ $\angle\ BSD = \angle\ CSD = \angle\ BAC$.
   (*b*) $\angle\ BSE$ is supplementary to $\angle\ BSD$,
   $\angle\ CSE$ is supplementary to $\angle\ CSD$,
   and $\angle\ ABC + \angle\ ACB$ is supplementary to $\angle\ BAC$;
   $\therefore$ $\angle\ BSE = \angle\ CSE = \angle\ ABC + \angle\ ACB$.
   (*c*) $\quad \angle\ ASC - \angle\ ASB = 2\ \angle\ ASE$;
   $\therefore 2\ \angle\ ABC - 2\ \angle\ ACB = 2\ \angle\ ASE$;
   $\therefore \quad \angle\ ABC - \angle\ ACB = \angle\ ASE$.

(*d*) Because arc BD = arc CD, $\therefore \angle BAD = \angle CAD$.
Because $\angle DAE$ is right (III. 31), $\therefore AE$ is $\perp AD$.
Now AD bisects $\angle BAC$;
$\therefore AE$ bisects the angle supplementary to $\angle BAC$, that is, the exterior vertical angle at A.

8. For that angle $= \angle ASE$ (I. 29) $= \angle ABC - \angle ACB$.

9. Let $\triangle ABC$ (fig. to IV. 5) be equilateral.
Then $SK^2 = SA^2 - AK^2 = SA^2 - AL^2 = SL^2$;
$\therefore SK = SL$.
Hence also $SL = SH$.

10. Let $\triangle ABC$ (fig. to IV. 5) be isosceles, and let $AB = AC$.
Then $SK^2 = SA^2 - AK^2 = SA^2 - AL^2 = SL^2$;
$\therefore SK = SL$.

11. Let ABC be a triangle, S the common centre of the inscribed and circumscribed circles.
Draw SD, SE, SF respectively $\perp BC$, CA, AB, and join SA, SB, SC.
Because S is the inscribed centre, $\angle SAE = \angle SAF$;
$\therefore \triangle$s SAE, SAF are congruent, and $AE = AF$. *I.* 26
But because S is the circumscribed centre,
$AE = \frac{1}{2} AC$, and $AF = \frac{1}{2} AB$; $\therefore AC = AB$.
Hence also $AB = BC$, and $\triangle ABC$ is equilateral.

12. Let ABC be a triangle, S the circumscribed, and I the inscribed centre; if A, S, I be collinear, then $AB = AC$.
Draw SK, SL respectively $\perp CA$, AB.
Since I is the inscribed centre, and ASI is a straight line,
$\therefore \angle SAK = \angle SAL$;
$\therefore \triangle$s SAK, SAL are congruent, and $AK = AL$. *I.* 26
Now $AK = \frac{1}{2} AC$, and $AL = \frac{1}{2} AB$;
$\therefore AC = AB$.

13. Because SH is $\perp BC$, and KL is $\parallel BC$, *App. I.* 1
$\therefore SH$ is $\perp KL$.
Hence also SK is $\perp LH$, and $SL \perp HK$;
$\therefore S$ is the orthocentre of $\triangle HKL$.

14. Let XYZ be the orthocentric triangle of $\triangle ABC$, and let SA meet YZ at D'.
From S draw $SK \perp AC$, and therefore bisecting AC and $\angle ASC$.
Then $\angle ASK = \frac{1}{2} \angle ASC = \angle ABC$. *III.* 20
But since the points B, C, Y, Z are concyclic, by the tenth deduction from III. 22,

$\therefore \angle AYD' = \angle ABC$; *III. 22, Cor.*
$\therefore \angle ASK = \angle AYD'$.
Now $\angle SAK = \angle YAD'$; $\therefore \angle AKS = \angle AD'Y$;
$\therefore \angle AD'Y$ is right.
Similarly SB, SC are respectively $\perp ZX$, XY.

15. Since (fig. to preceding deduction),
$\angle YAD'$ = complement of $\angle AYD'$,
= complement of $\angle ABC$,
$= \angle BAX$;
$\therefore$ the bisector of $\angle BAC$ must be also the bisector of $\angle SAX$,
and the bisector of $\angle BAC$ passes through the inscribed centre.

PROPOSITION 6.

1. Because AC is $\perp BD$,
$\therefore$ $\angle$s AOB, BOC, COD, DOA are all equal;
$\therefore$ arcs AB, BC, CD, DA are all equal;
$\therefore$ chords AB, BC, CD, DA are all equal.
2. For $AB^2 = AO^2 + BO^2 = 2\,AO^2$;
and $AC^2 = AB^2 + BC^2 = 2\,AB^2$.
3. For they are all double the square on the radius.
4. The quadrilateral thus inscribed has its diagonals bisecting each other;
$\therefore$ it is a $\|^m$, by the ninth deduction from I. 27.
And the diagonals of this $\|^m$ are equal;
$\therefore$ it is a rectangle, by the seventh deduction from I. 34.
5. Angle AOB is right.
6. $AB^2 = AO^2 + BO^2 = r^2 + r^2$; $\therefore AB^2 = 2\,r^2$;
$\therefore AB = r\sqrt{2}$.

PROPOSITION 7.

1. If the tangents at A and B do not meet, they are parallel;
$\therefore$ OA and OB, which are respectively $\perp$ these tangents, are parallel, which is absurd.
Similarly for the other pairs of tangents.

2. The square circumscribed about a circle
   = the square on a diameter of the circle
   = twice the square inscribed in the circle,
   by the second deduction from IV. 6.
3. For they are all equal to the square on the diameter.
4. By the seventh deduction from III. 17, if a $\|^m$ be circumscribed about a circle, it must be a rhombus. Now if the $\|^m$ have its angles right, the angles of the rhombus must be right, that is, the rhombus is a square.
5. Tangents at the ends of a diameter are parallel, by the first deduction from III. 18;
   $\therefore$ this circumscribed figure is a $\|^m$; and by the seventh deduction from III. 17, it is a rhombus.
6. Angle EOF may be proved to be right.
7. If $OA = r$, then $FG = AC = 2r$.

PROPOSITION 8.

1. Bisect AB and BC at E and F; through E and F draw EG and FH respectively $\|$ BC and AB.
2. If a circle could be inscribed in a rectangle which is not a square, then a rectangle which is not a square could be circumscribed about a circle, which is contrary to the fourth deduction from IV. 7.
3. Repeat the construction and proof of the proposition, substituting the word 'rhombus' for 'square.'
4. A square, a rhombus.
5. For $OF = \frac{1}{2} AB = \frac{1}{2} a$.

PROPOSITION 9.

1. By the first deduction from III. 22, a circle cannot be circumscribed about a $\|^m$, unless it be a rectangle. If the $\|^m$ therefore have its sides all equal, it must be a square.
2. Draw the two diagonals, which are equal, and bisect one another. Their point of intersection is the centre, and half of either diagonal the radius, of the required circle.
3. A square, a rectangle.

4. For $AB^2 = AO^2 + BO^2 = 2\,AO^2$;
$\therefore\ a^2 = 2\,AO^2$, and $2\,a^2 = 4\,AO^2$;
$\therefore\ 2\,AO = a\sqrt{2}$, and $AO = \frac{1}{2}\,a\sqrt{2}$.

PROPOSITION 10.

1. For $\angle\,B$ is common, and $\angle\,BDC = \angle\,A$; *III.* 32
2. For $\angle\,A + \angle\,B + \angle\,ADB = \angle\,A$ + twice $\angle\,A$ + twice $\angle\,A$
$= 5$ times $\angle\,A$;
and $\angle\,A + \angle\,B + \angle\,ADB = 2$ rt. $\angle$s;
$\therefore$ 5 times $\angle\,A = 2$ rt. $\angle$s.
3. Make an isosceles triangle such as ABD, and bisect $\angle\,A$.
Then at the vertex of the right angle make an angle equal to half of $\angle\,A$.
4. For $\angle\,ACD = \angle\,B + \angle\,BDC =$ twice $\angle\,A + \angle\,A$
$=$ thrice $\angle\,A$,
and $\angle\,CDA = \angle\,A$.
5. Make any isosceles triangle such as ABD, and on the given base construct a triangle equiangular to $\triangle\,ABD$.
6. On the given base construct a triangle equiangular to $\triangle\,ACD$.
7. If the small circle do not cut the large one, it must touch it at D. Now BD is a tangent to the small circle at D;
$\therefore$ it must be a tangent to the large circle at D, by the third deduction from III. 16.
But BD is a chord of the large circle.
8. For $\angle\,ADB = \angle\,AFD$ (III. 32) $= \angle\,ADF$; *I.* 5
$\therefore\ \angle\,BAD = \angle\,FAD$, and $\triangle$s BAD, FAD are congruent;
$\therefore\ DF = BD$.
9. For $\angle\,BAD =$ one-fifth of 2 rt. $\angle$s,
$=$ one-tenth of 4 rt. $\angle$s.
10. For $\angle\,CAD =$ one-fifth of 2 rt. $\angle$s;
$\therefore$ the angle at the centre of the small circle subtended by $CD =$ one-fifth of 4 rt. $\angle$s; and $CD = AC$.
11. This follows from the second deduction from III. 24.
Or, the small circle is circumscribed about $\triangle\,AFD$, which in the eighth deduction was shown to be congruent to $\triangle\,ABD$.
12. For $\angle\,BAF =$ twice $\angle\,BAD =$ one-fifth of 4 rt. $\angle$s.
13. It was proved in the eighth deduction that $FD = BD$;
$\therefore\ FD = DC$ or CA; $\therefore$ arc $FDC =$ arc DCA;
$\therefore$ chord $FC =$ chord DA.

Now $FA = AB$, and $AC = BD$;

$\therefore$ $\triangle$s FAC, ABD are congruent.

14. For $\angle BDG$ = one-fifth of 2 rt. $\angle$s;

$\therefore$ $\angle BAG$ = one-fifth of 4 rt. $\angle$s. *III.* 20

15. By the previous deduction $\angle CAG$ = one-fifth of 4 rt. $\angle$s;

and $\angle ACG = \angle BCD$ = one-fifth of 4 rt. $\angle$s;

$\therefore$ $GA = GC$. *I.* 6

Hence $FAGC$ is a rhombus;

$\therefore$ FG bisects AC perpendicularly.

16. Since $\angle BAD$ = one-fifth of 2 rt. $\angle$s = $\frac{2}{5}$ of a right angle, and since a right angle can be trisected, by the tenth deduction from I. 32;

$\therefore$ an angle can be obtained = $(\frac{2}{5} - \frac{1}{3})$ of a right angle.

Now $\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$.

17. Join FA, FC, FB.

Since $\triangle FAC$ is isosceles,

$\therefore$ $FB^2 - FA^2 = AB \cdot BC$, by first deduction of Book II.,

$= BD^2$.

Now FB is a side of a regular pentagon, and BD a side of a regular decagon inscribed in the large circle.

18. After dividing AB internally at C so that $AB \cdot BC = AC^2$, construct on AB a triangle ADB such that AD shall $= AB$, and $BD = AC$. *I.* 22

19. Let DH be $\perp$ BC.

Then $AD^2 = AB^2 + BD^2 - 2\,AB \cdot BH$. *II.* 13

Now $AD^2 = AB^2$; $\therefore$ $BD^2 = 2\,AB \cdot BH$;

$\therefore$ $AB \cdot BC = 2\,AB \cdot BH$; $\therefore$ $BC = 2\,BH$.

Hence $\triangle DBC$, and consequently $\triangle CAB$, is isosceles.

The rest of the proof is the same as in the proposition.

20. Let O be the middle point of the arc CD. Join DO and produce it to meet BC at H.

Because arc $CO = \frac{1}{2}$ arc CD.

$\therefore$ $\angle CDO = \frac{1}{2} \angle CAD$ *III.* 27, 21

$= \frac{1}{2} \angle CDB$;

$\therefore$ DH bisects $\angle CDB$;

$\therefore$ DH bisects BC perpendicularly.

Again if DC be bisected at K, and a perpendicular to DC from K be drawn, it will pass through O. *III.* 30

Hence O is the circumscribed centre of $\triangle DBC$.

21. See the ninth deduction.

22. If $AB = r$, then $AC = \frac{1}{2} r (\sqrt{5} - 1)$ by the Algebraical Application of II. 11.
$\therefore BD = \frac{1}{2} r (\sqrt{5} - 1)$.

PROPOSITION 11.

1. Two can be drawn from A;
$\therefore$ two can be drawn from every vertex, which would give 10. But it will be found that in this manner every diagonal is drawn twice over.
$\therefore$ the number of diagonals is 5.
2. Because $\angle BAC = \angle ACE$ (III. 27); $\therefore AB$ is $\parallel CE$.
3. For $\triangle$s ABC, BCD are congruent (I. 4); $\therefore AC = BD$.
4. Let AC, BD cut each other at E'.
Then since $\triangle ACD$ has each of its base angles double of the vertical angle CAD, and since $\angle ADC$ is bisected by DE', we have the same species of figure as in IV. 10;
and $\therefore AC \cdot CE' = AE'^2$.
Hence also the other diagonals are cut in medial section.
5. Let AC, BD intersect at E'; BD, CE at A'; CE, DA at B'; DA, EB at C'; EB, AC at D'.
Since the diagonals are equal and all divided internally in medial section.
$\therefore$ the segments AC', AD', BD', BE', CE', CA', DA', DB', EB', EC' are all equal.
Hence $CE - (CA' + EB') = DA - (DB' + AC')$;
$\therefore A'B' = B'C'$.
Similarly for the other pairs of sides.
Since CE is $\parallel AB$ and $DB \parallel AE$,
$\therefore \angle E'A'B' = \angle EAB$, *I. 34, Cor.*
= an angle of a regular pentagon.
Similarly for the other angles.
6. Let $A'B'C'D'E'$ (fig. to preceding deduction) be the given regular pentagon; by producing its alternate sides to meet, let the figure $ABCDE$ be obtained.
Then $\triangle$s $AC'D'$, $BD'E'$, $EB'C'$ are isosceles, *I.* 13, 6
and congruent to each other; *I.* 26
$\therefore \triangle$s ABD', AEC' are congruent, and $AB = AE$. *I.* 15, 4
Similarly for the other pairs of sides of $ABCDE$.

Since (see the eleventh deduction following) an angle of a regular pentagon = $\frac{6}{5}$ of a right angle,

$\angle$s BAD', $D'AC'$, EAC' may each be proved = $\frac{2}{5}$ of a right angle;

$\therefore$ $\angle$ BAE = an angle of a regular pentagon.

Similarly for the other angles of $ABCDE$.

7. The five pentagons are $A'BCDE$, $B'CDEA$, $C'DEAB$, $D'EABC$, $E'ABCD$.

Since $\triangle$ ACD with $\angle$ ADC bisected by DE' is the same species of figure as in IV. 10, $E'A = E'D = DC = CB = BA$;

$\therefore$ pentagon $E'ABCD$ is equilateral.

It is easily seen that it is not equiangular.

Hence also the other pentagons are equilateral and not equiangular.

8. Since $ABCB'$ is a $\|^{m}$, $\therefore$ $\triangle$ $ABC = \triangle$ $AB'C$; *I. 34*

and $\triangle$ $ABC = \triangle$ AED; *I. 4*

$\therefore$ the figure $ABCB'DE$, which is less than the pentagon $ABCDE$, = three times $\triangle$ ABC.

Four times $\triangle$ ABC = the figure $ABCB'DE$ + $\triangle$ CDE, which is greater than the pentagon $ABCDE$ by $\triangle$ $B'DE$.

9. Because $\triangle$ $ACD = \triangle$ $AB'C$ + $\triangle$ CDB',

$\therefore$ twice $\triangle$ $ACD = \|^{m}$ $ABCB'$ + $\triangle$ CDB' + $\triangle$ AEB',

which is less than the pentagon $ABCDE$ by $\triangle$ $B'DE$.

Because $\triangle$ ACD is greater than $\triangle$ ABC or $\triangle$ AED,

$\therefore$ 3 $\triangle$ ACD is greater than $\triangle$ ABC + $\triangle$ ACD + $\triangle$ AED, which is equal to the pentagon $ABCDE$.

10. Let BDE (fig. to IV. 10) be the given circle.

Draw any radius AB; divide it internally in medial section at C, so that AC is the greater segment.

Place in the circle two chords BD, DF each = AC;

then BF is a side of the inscribed regular pentagon.

11. By I. 32, Cor. 3, the five angles = 6 rt. $\angle$s.

Now the five angles are all equal;

$\therefore$ each angle = $\frac{6}{5}$ of a right angle.

12. Let AB be the given straight line.

At B make $\angle$ $ABC = \frac{4}{5}$ of a right angle, and cut off $BC = AB$;

at C make $\angle$ $BCD = \frac{4}{5}$ of a right angle, and cut off $CD = BC$;

at D make $\angle$ $CDE = \frac{4}{5}$ of a right angle, and cut off $DE = CD$.

Join EA.

13. Let CD (fig. to the proposition; the circle is not required) be the given straight line.

A regular pentagon could be constructed on CD, if the vertex A of the isosceles $\triangle ACD$ could be found, that is, if the length of AC could be found. Now if AC, BD intersect at E', then $AE' = ED$, by the second deduction and I. 34, that is $= CD$.

But if AC be divided internally in medial section, AE' is the greater segment; therefore the problem of finding A is reduced to this problem:

Given the greater segment of a straight line divided internally in medial section, to find the straight line.

This problem is solved in the first part of II. 11; for there AB (fig. to II. 11) is given, and a straight line CF is found such that $CF \cdot FA = AB^2$ or AC^2. Hence the construction for describing a regular pentagon on CD is as follows:

From D draw $DL \perp CD$, and $= \frac{1}{2} CD$;

with L as centre and LD as radius describe a circle, and produce CL to meet this circle at M.

With C and D as centres, and a radius equal to CM, describe arcs cutting each other in A. With A and C as centres, and a radius equal to CD, describe arcs cutting each other in B; and with A and D as centres, and a radius equal to CD, describe arcs cutting each other in E.

$ABCDE$ is the required regular pentagon.

14. Let $A'B'C'D'E'$ (fig. to the fifth deduction) be the given regular pentagon.

By the sixth deduction, $\angle D'AC' = \frac{2}{5}$ of a right angle;

$\therefore$ the sum of the five angles such as $\angle D'AC' = 2$ rt. $\angle$s.

15. One-fifth of four right angles.

16. Let p denote the side of a regular pentagon, d the side of a regular decagon, inscribed in the same circle, whose radius is r.

Then $p^2 = r^2 + d^2$, by the seventh deduction from IV. 10,

$$= r^2 + \left\{ \tfrac{1}{2} r (\sqrt{5} - 1) \right\}^2,$$

$$= \frac{r^2 (10 - 2\sqrt{5})}{4};$$

$$\therefore \qquad p = \tfrac{1}{2} r \sqrt{10 - 2\sqrt{5}}.$$

PROPOSITION 12.

1. If the tangents at B and C do not meet, they are parallel, and consequently OB and OC, which are perpendicular to these tangents, are also parallel, which is absurd.
   Hence also every pair of consecutive tangents meet.
2. Let $ABCDE$ be a regular pentagon inscribed in a circle whose centre is O. Bisect arc AB at G; at G draw $A'B' \parallel AB$, and meeting OA, OB produced at A', B'; through B' draw $B'C' \parallel BC$, and meeting OC produced at C'. Join OG, and draw $OH \perp B'C'$.
   Since $A'B'$ is $\parallel AB$, and $\triangle\ OAB$ is isosceles,
   $\therefore\ \triangle\ OA'B'$ is also isosceles.
   Similarly $\triangle\ OB'C'$ is isosceles.
   But since $\triangle$s $OA'B'$, $OB'C'$ have one of the equal sides common, and the angles at the vertex O equal,
   $\therefore$ they are congruent, and $A'B' = B'C'$. *I.* 4
   Again since G is the middle point of arc AB,
   $\therefore\ OG$ is $\perp AB$, and consequently to $A'B'$;
   $\therefore\ A'B'$ is a tangent to the circle. *III.* 16
   By comparing the right-angled $\triangle$s OHB', OGB', it may be proved that $OH = OG$, *I.* 26
   $= \text{a radius}$;
   $\therefore\ B'C'$ is a tangent to the circle. *III.* 16
   Lastly $\angle\ A'B'C' = \angle\ ABC$, *I.* 34, *Cor.*
   $=$ an angle of a regular pentagon.
3. One-fifth of four right angles.
4. The proof of this deduction is the proof of the proposition with hardly any other changes than the substitution of the word 'polygon' for 'pentagon.'

PROPOSITION 13.

1. From the proof of the proposition it is seen that the centre of the circle inscribed in a regular pentagon is situated on the straight line which bisects any side perpendicularly. Now since $\triangle\ ACD$ is isosceles, by the third deduction from IV. 11, the straight line which bisects CD perpendicularly passes through A.

Hence the centre of the circle may be found by drawing from any two vertices perpendiculars to the opposite sides. Their point of intersection is the centre. The radius of the circle is the distance from the centre to the foot of one of the perpendiculars.

2. Area of $\triangle\ OCD \quad = \frac{1}{2}\ CD \cdot OG$;

$\therefore$ area of $ABCDE = 5$ times $\triangle\ OCD$,

$= \frac{1}{2}$ (5 times $CD \cdot OG$),

$=$ semiperimeter $\cdot$ inscribed radius.

PROPOSITION 14.

1. Since $\triangle\ OCD$ is isosceles, O is situated on the straight line which bisects CD perpendicularly.
   Hence the centre of the circumscribed circle is found in the same way as the centre of the inscribed circle.
   See the first deduction from IV. 13.

   The radius of the circumscribed circle is the distance from the centre to any one of the vertices of the regular pentagon.

2. In the fig. to IV. 13, OC is a radius of the circle circumscribed about the regular pentagon, OG a radius of the circle inscribed in the regular pentagon, and CG is half of a side of the regular pentagon.

   Now $\quad OC^2 = CG^2 + OG^2$; *I.* 47

   $\therefore \quad 4\ OC^2 = 4\ CG^2 + 4\ OG^2$;

   $\therefore \quad (2\ OC)^2 = (2\ CG)^2 + (2\ OG)^2$.

3. By the sixteenth deduction from IV. 11, when a side of a regular pentagon is $\frac{1}{2}\ r \sqrt{10 - 2\sqrt{5}}$, the radius of the circumscribed circle is r. What, then, will the radius of the circumscribed circle be, when a side of the regular pentagon is a?

   $\frac{1}{2}\ r \sqrt{10 - 2\sqrt{5}} : r :: a :$ required radius.

   $$\text{Radius} = \frac{ar}{\frac{1}{2}\ r\sqrt{10 - 2\sqrt{5}}},$$

   $$= \frac{2\ a}{\sqrt{10 - 2\sqrt{5}}};$$

$$\therefore \text{radius}^2 = \frac{4a^2}{10 - 2\sqrt{5}},$$

$$= \frac{4a^2(10 + 2\sqrt{5})}{(10 - 2\sqrt{5})(10 + 2\sqrt{5})},$$

$$= \frac{4a^2(10 + 2\sqrt{5})}{100 - 20},$$

$$= \frac{a^2(10 + 2\sqrt{5})}{20},$$

$$= \frac{a^2(50 + 10\sqrt{5})}{100};$$

$$\therefore \text{radius} = \tfrac{1}{10}\, a \sqrt{50 + 10\sqrt{5}}.$$

PROPOSITION 15.

1. For $\angle$s EAC, ACE, CEA stand each on an arc equal to $\frac{1}{3}$ $\bigcirc^{ce}$;
   $\therefore$ each of these angles $= \frac{1}{3}$ of 2 rt. $\angle$s;
   $\therefore$ $\triangle$ ACE is equiangular, and consequently equilateral.
2. For $\triangle\, ACE = \triangle\, AOC + \triangle\, COE + \triangle\, EOA$,
   $= \frac{1}{2}\, AOCB + \frac{1}{2}\, COED + \frac{1}{2}\, EOAF$,
   $= \frac{1}{2}\, ABCDF$.
3. Let AB be the given straight line.
   On AB describe the equilateral $\triangle$ OAB; *I.* 1
   with O as centre, and OA as radius, describe a circle, in which place chords BC, CD, DE, EF, FA each equal to AB.
   $ABCDEF$ is the required hexagon.
   The proof follows from the Cor. to the proposition.
4. This deduction follows from the fact that every regular hexagon $ABCDEF$ is made up of six equilateral triangles, with their vertices joined at a point O.
5. Let $ABCDEF$ be a regular hexagon.
   Join FB, EC.
   Then $\triangle$s FAB, EDC are congruent, and $FB = EC$; *I.* 4
   $\therefore$ $FBCE$ is a $\|^{m}$, by the fifth deduction from I. 34.
   $\therefore$ FE is $\|$ BC.
   Similarly for the other pairs of opposite sides.

6. Since $FBCE$ is a $\|^m$, by the preceding deduction,
$\therefore$ FC passes through the middle point of BE, by the tenth deduction from I. 29.
Similarly, since $ABDE$ is a $\|^m$, AD passes through the middle point of BE;
$\therefore$ AD, BE, CF are concurrent.
Again, since the quadrilateral $ABCD$ is congruent to the quadrilateral $AFED$, by the thirteenth deduction from I. 4,
$\therefore$ AD bisects $\angle\ FAB$;
$\therefore$ $\angle\ BAD = \frac{1}{3}$ of 2 rt. $\angle$s.
But $\angle\ ABC = \frac{2}{3}$ of 2 rt. $\angle$s;
$\therefore$ AD is $\|$ BC. *I.* 28
7. Three can be drawn from A;
$\therefore$ three can be drawn from every vertex, which would give 18. But it will be found that in this manner every diagonal is drawn twice over;
$\therefore$ the number of diagonals is 9.
8. In the fifth deduction it was proved that $FBCE$ is a $\|^m$;
$\therefore$ BF is $\|$ CE.
The other pairs of parallel diagonals are AE, BD, and AC, FD.
9. This follows from the second part of the fifth deduction from IV. 3, and from the second deduction from IV. 15.
10. From E (fig. to the proposition) draw $EG \perp OD$, and join AE.
Then AE is a side of the inscribed equilateral triangle;
AO and EO are each $= AB$; and $OG = \frac{1}{2}\ OE$, by the eighth deduction from I. 9.

$$\begin{aligned} \text{Hence } AE^2 &= AO^2 + EO^2 + 2\,AO \cdot OG, \qquad \textit{II. } 12 \\ &= AO^2 + EO^2 + AO \cdot OE, \\ &= 3\,AB^2. \end{aligned}$$

11. One-sixth of four right angles, or one-third of two right angles.
12. Let $ABCDEF$ be a regular hexagon.
Bisect any two consecutive angles ABC, BCD by BO, CO which meet at O. O is the centre both of the circle inscribed in the hexagon, and of the circle circumscribed about it. The radius of the inscribed circle is the perpendicular from O to BC; the radius of the circumscribed circle is OB or OC.
To circumscribe a regular hexagon about a circle, inscribe a regular hexagon in the circle, and at the vertices draw tangents to the circle.

PROPOSITION 16.

1. Since arc $AC = \frac{1}{3}$ of the $\bigcirc^{ce}$,
   and arc $AB = \frac{1}{5}$ of the $\bigcirc^{ce}$;
   $\therefore$ arc $BC = (\frac{1}{3} - \frac{1}{5})$, or $\frac{2}{15}$, of the $\bigcirc^{ce}$.
   Bisect arc BC in F; *III.* 30
   BF will be a side of an inscribed regular quindecagon.
2. The arc cut off by a side of an inscribed regular hexagon $= \frac{1}{6}$ of the $\bigcirc^{ce}$; the arc cut off by a side of an inscribed regular decagon $= \frac{1}{10}$ of the $\bigcirc^{ce}$;
   $\therefore$ the difference of these arcs $= (\frac{1}{6} - \frac{1}{10})$, or $\frac{1}{15}$, of the $\bigcirc^{ce}$.
   Hence the chord which cuts off an arc equal to this difference is a side of an inscribed regular quindecagon.
3. Let 1, 2, 3, 4, ... 15 denote the vertices of a regular quindecagon inscribed in the circle. Draw straight lines from 1 to 3, from 3 to 8, and from 8 to 1, cutting off arcs which are to one another as the numbers 2, 5, 8. The angles which stand on these arcs will also be to one another as the numbers 2, 5, 8.

   Draw straight lines from 1 to 5, from 5 to 10, and from 10 to 1.
4. Bisect any two consecutive angles of the regular quindecagon. The point in which the bisectors meet will be the centre both of the circle inscribed in the quindecagon, and of the circle circumscribed about it. The radius of the inscribed circle is the perpendicular from the centre on any one of the sides; the radius of the circumscribed circle is the distance from the centre to any one of the vertices of the quindecagon.

   To circumscribe a regular quindecagon about a circle, inscribe a regular quindecagon in the circle, and at the vertices draw tangents to the circle.
5. Twelve can be drawn from one vertex;
   $\therefore$ twelve can be drawn from every vertex, which would give 180. But it will be found that in this manner every diagonal is drawn twice over;
   $\therefore$ the number of diagonals is 90.
6. Fix upon A, one of the n vertices of the polygon; there will remain $n - 1$ other vertices, and therefore there can be drawn from A to these vertices $n - 1$ straight lines. But of these $n - 1$ straight lines only $n - 3$ will be diagonals, since the

other two will be consecutive sides of the polygon. Hence from each of the n vertices of the polygon there can be drawn $n - 3$ diagonals, which would give $n (n - 3)$ diagonals. But it will be found that in this manner every diagonal is drawn twice over; therefore the number of diagonals is $\frac{1}{2} n (n - 3)$.

7. This has been proved in propositions 13 and 14 for a regular pentagon, and no properties of a regular pentagon have been assumed which are not common to it and all regular polygons. Hence this method of finding the centres holds good for all regular polygons.

DEDUCTIONS.

1. Let $ABCD \ldots . N$ be an equilateral n-gon inscribed in a circle. Then $\angle$ s $NAB, ABC, BCD, \ldots .$ stand on arcs which are each $\frac{n-2}{n}$ths of the $\bigcirc^{ce}$; $\therefore$ these angles are equal.

2. (*a*) Draw any radius OA of the circle; at O the centre make $\angle\ AOB = \frac{1}{3}$ of 4 rt. $\angle$ s. Then if a circle can be inscribed in the sector OAB, it will be one of the three equal circles required.

Bisect $\angle\ AOB$ by OC, which meets the $\bigcirc^{ce}$ at C; at C draw a tangent DCE, meeting OA, OB produced at D, E. In $\triangle\ ODE$ inscribe a circle. This will be the circle inscribed in the sector OAB.

For the circle inscribed in the sector OAB must, in order to touch OA, OB have its centre on OC (see the fifth example of Loci, Book III.), and, in order to touch the arc ACB, must pass through C (III. 11). But since $\triangle\ ODE$ is isosceles (I. 26), the circle inscribed in it must have its centre on OC, and touch DE at C. Hence the two circles are identical.

(*b*) Draw any radius OA; at O the centre make $\angle\ AOB =$ a right angle. Inscribe a circle in the sector OAB, by the method preceding. This circle will be one of the four equal circles required.

(*c*) Make a sector whose angle is $\frac{1}{5}$ of 4 rt. $\angle$ s, by the third deduction from IV. 10; and inscribe a circle in it. This circle will be one of the five equal circles required.

(*d*) Make a sector whose angle is $\frac{1}{6}$ of 4 rt. $\angle$s, and inscribe a circle in it. This circle will be one of the six equal circles required.

3. Let ABC be an equilateral triangle, and from A let AD be drawn $\perp$ BC; on BC as diameter let a circle be described: to prove AD = a side of the equilateral triangle inscribed in the circle.

Bisect $\angle$ ABC by BE, meeting the $\bigcirc^{ce}$ at E, and join CE.

Then $\angle$ $EBC = \frac{1}{6}$ of 2 rt. $\angle$s, and $\angle$ BEC is right; *III.* 31

$\therefore$ $\angle$ $BCE = \frac{1}{3}$ of 2 rt. $\angle$s; *I.* 32

$\therefore$ BE is a side of an equilateral triangle inscribed in the circle.

Now $AD = BE$, since $\triangle$s ABD, BCE are congruent. *I.* 26

4. Let $ABCDEF$ (fig. to IV. 15) be a regular hexagon inscribed in a circle; at A and B draw two tangents AG, BG meeting at G.

Then $\triangle$ $OAB = \frac{1}{6}$ of the area of the inscribed regular hexagon,

and $OAGB = \frac{1}{6}$ of the area of the circumscribed regular hexagon.

Because $\angle$s OAG, OBG are right, *III.* 18

and $\angle$s OAB, OBA are each $\frac{2}{3}$ of a right angle;

$\therefore$ $\angle$s BAG, ABG are each $\frac{1}{3}$ of a right angle.

Hence if $\triangle$ GAB were rotated round AB through two right angles, G would occupy the position of the centroid of $\triangle$ OAB, since $\triangle$ OAB is equilateral;

$\therefore$ $\triangle$ $GAB = \frac{1}{3}$ $\triangle$ OAB;

$\therefore$ $\triangle$ $OAB = \frac{3}{4}$ $OAGB$;

$\therefore$ area of $ABCDEF = \frac{3}{4}$ area of the circumscribed regular hexagon.

5. Let ABC be the given equilateral triangle.

Bisect $\angle$s ABC, ACB by BO, CO meeting at O;
through O draw $EH \parallel BC$, meeting AB, AC at E, H;
through O draw $GD \parallel CA$, meeting BC, BA at G, D;
through O draw $KF \parallel AB$, meeting CA, CB at K, F.
Join EF, GH, KD. $DEFGHK$ is the required hexagon.

For $\triangle$ OFG is equiangular to $\triangle$ ABC; *I.* 29, 32

$\therefore$ $\triangle$ OFG is equilateral. *I.* 6, *Cor.*

Similarly $\triangle$s OHK, ODE are equilateral.

Again, $\triangle$ OEF is equiangular to $\triangle$ ABC; *I.* 34, *Cor.*

$\therefore$ $\triangle$ OEF is equilateral. *I.* 6, *Cor.*

Similarly $\triangle$s OGH, OKD are equilateral.
Hence $DEFGHK$ is a regular hexagon.

The area of the hexagon is two-thirds of the area of $\triangle\ ABC$, since the former consists of six equal equilateral triangles, and the latter of nine.

6. Let $ABCDEF$ (fig. to IV. 15) be a regular hexagon inscribed in the given circle. The regular dodecagon is derived from it by bisecting the arcs AB, BC, &c.

Let G be the middle point of the arc AB; join AG, GO, and let GO meet AB at H.

Then area of the dodecagon $= 12\ \triangle\ AGO$,
$= 6\ AH \cdot GO$,
$= 3\ AB \cdot GO$,
$= 3\ AB^2$,
$= AE^2$, by the tenth deduction from IV. 15.

7. Since a regular octagon can be built up of eight equal isosceles triangles with their vertices joined at one point; the problem will be solved if we can construct on AB, the given straight line, as base, one such isosceles triangle.
Now each of the vertical angles of these isosceles triangles is half a right angle, and therefore each of the base angles three-fourths of a right angle. The problem is thus reduced to the tenth deduction from I. 23.

8. Let $ABCD$ (fig. to IV. 8) be the circumscribed square, and let OA meet the $\bigcirc^{ce}$ at K, and the straight line EH at L.

Then EH is a side of the inscribed square, and EK a side of the inscribed regular octagon.
Now the octagon $= 8\ \triangle\ EOK$,
$= 4\ EL \cdot OK$,
$= 2\ EL \cdot 2\ OK$,
$= EH \cdot AD$.

9. Because $EF^2 = ED^2 = EC^2 + CD^2 = EC^2 + CB^2$;
and $EF^2 = EC^2 + CF^2 + 2\ EC \cdot CF$, *II.* 4
$= EC^2 + CF^2 + AC \cdot CF$,
$= EC^2 + CF^2 + BC \cdot CF$;
$\therefore\ EC^2 + CF^2 + BC \cdot CF = EC^2 + CB^2$;
$\therefore\ CF^2 = BC^2 - BC \cdot CF$,
$= BC \cdot BF$;
$\therefore\ BC$ is cut in medial section at F, and CF is the greater segment.

[This result follows much more simply if the given construction is compared with that of II. 11.]

Hence, by the ninth deduction from IV. 10, CF is the side of a regular decagon inscribed in the circle.

Again, since CD is a side of a regular hexagon inscribed in the circle, and CF a side of a regular decagon;

$\therefore$ DF is a side of the inscribed regular pentagon, by the seventeenth deduction from IV. 10.

10. Let $ABCDE$ be a regular pentagon, BE, CE two of its diagonals, which are equal to each other, and parallel respectively to CD, AB, by the third and second deductions from IV. 11: to prove that, if the trapezium $ABCE$ be rotated round BC through two right angles, it will fall exactly within EB and DC produced.

Bisect BC at F; join EF and produce it its own length to E', and join $E'C$.

Since $\triangle$ EBC is isosceles, EF is $\perp$ BC;

$\therefore$ after the rotation of the trapezium $ABCE$ round BC, EF will occupy the position $E'F$, and EC that of $E'C$.

Now since $\angle\ BEC = \frac{2}{5}$ of a right angle,

$\therefore$ $\angle\ FEC = \frac{1}{5}$ of a right angle;

$\therefore$ $\angle\ FE'C = \frac{1}{5}$ of a right angle.

But $\angle\ E'FC =$ a right angle;

$\therefore$ $\angle\ E'CF = \frac{4}{5}$ of a right angle. *I.* 32

And $\angle\ BCD = \frac{6}{5}$ of a right angle, by the eleventh deduction from IV. 11;

$\therefore$ $\angle\ E'CF + \angle\ BCD = 2$ rt. $\angle$s;

$\therefore$ $E'C$ and CD are in one straight line, *I.* 14

that is, after rotation E falls on DC produced.

Again, since AB is $\parallel$ CE, $A'B$, the position of AB after rotation, will be $\parallel$ CE', the position of CE after rotation.

But CE' is in the same straight line as CD;

$\therefore$ $A'B$ must be in the same straight line as BE, which is $\parallel$ CD.

Similarly for the trapezium $AEDB$.

11. Let BD (fig. to IV. 10) be the given straight line.

Find a straight line such that when it is divided internally in medial section its greater segment shall be equal to BD.

This is done in the thirteenth deduction from IV. 11.

On BD as base construct an isosceles $\triangle$ ABD having AB and AD equal to this straight line. With A as centre, and

AB or AD as radius, describe the circle BDE; and in it place successively chords equal to BD.

The proof follows from the ninth deduction from IV. 10.

12. (a) Let $ABCD$ be the given square, E the middle of AB.

With E as centre, and radius equal to a side of the square, cut AD, BC at F and G. Join EF, FG, GE.

EFG is the equilateral triangle required.

For $\triangle$s EAF, EBG are congruent, and $AF = BG$; *I. A, Cor.*

$\therefore FG = AB$, *I. 33, 34*

$= EF = EG$.

(b) Let $ABCD$ be the given square.

At A make $\angle$s BAE, DAF each $= \frac{1}{3}$ of a right angle, by the tenth deduction from I. 32, and I. 9; let AE, AF meet BC, CD at E, F; and join EF.

AEF is the equilateral triangle required.

For $\triangle$s ABE, ADF are congruent, and $AE = AF$; *I. 26*
and $\angle\ EAF = \frac{2}{3}$ of a right angle;

$\therefore \triangle\ AEF$ is equilateral, by the twelfth deduction from I. 32.

13. Let DEF, $D_1E_1F_1$ (fig. on p. 251 of *Euclid*) be the given circles. By the eighteenth deduction of Book III., draw AE_1, AF_1 direct common tangents to the circles, and BC one of the transverse common tangents. ABC is the required triangle.

14. Let $D_2E_2F_2$, $D_3E_3F_3$ be the given circles.

Draw F_2B, E_3C transverse common tangents to the circles, and BC one of the direct common tangents.

ABC is the required triangle.

15. From the twentieth deduction following it will be seen that the triangle required is the orthocentric triangle of the triangle formed by joining the three centres.

16. Draw two straight lines AE_1, AF_1 (fig. on p. 251 of *Euclid*) containing the given vertical angle; with the inscribed radius as radius describe a circle to touch AE_1, AF_1 at E and F.

To do this, from any points in AE_1, AF_1 draw two perpendiculars to AE_1, AF_1, and make them equal to the inscribed radius; through the ends of the perpendiculars draw parallels to AE_1, AF_1; the parallels will intersect at I, the centre of the required circle.

Cut off EE_1, FF_1 each $=$ the given base; then $AE_1 = AF_1$. Describe a circle to touch AE_1 AF_1 at E_1 and F_1.

Draw BC a transverse common tangent to these two circles.

ABC is the required triangle.

For BC or $a = EE_1 = FF_1$, by (3) of the nineteenth deduction following.

17. Draw two straight lines AE_1, AF_1 (fig. on p. 251 of *Euclid*) each = the semi-perimeter, and containing the given vertical angle. Describe a circle to touch AE_1, AF_1 at E_1, F_1; with the inscribed radius as radius describe a circle to touch AE_1 and AF_1; and to these two circles draw a transverse common tangent BC. ABC is the required triangle.

18. When the sum of the sides is given.

Since the base is given, and the sum of the sides is given,

$\therefore$ the perimeter is given;

$\therefore$ if the vertical angle could be found, the problem would be reduced to the preceding.

Suppose ABC (fig. on p. 251 of *Euclid*) to be the triangle sought.

$BC = BD + CD = BF + CE$; *III.* 17, *Cor.*

$\therefore AB + AC - BC = AF + AE = 2\,AF.$ *III.* 17, *Cor.*

Now $AB + AC - BC$ is known;

$\therefore AF$ is known;

$\therefore$ of the right-angled $\triangle\ AFI$, AF and IF are known;

$\therefore \angle\ IAF$ is known.

But $\angle\ IAF$ is half of the vertical angle, by the second deduction from III. 17;

$\therefore$ the vertical angle is known.

When the difference of the sides is given.

Draw BC (fig. on p. 251 of *Euclid*) = the given base; bisect BC at H, and cut off HD = half the given difference of sides. With the inscribed radius as radius describe a circle to touch BC at D; from B and C draw tangents to this circle meeting each other at A.

ABC is the required triangle.

For $AB \sim AC = (AF + BE) \sim (AE + CE)$,

$= BF \sim CE$, *III.* 17, *Cor.*

$= BD \sim CD$, *III.* 17, *Cor.*

$= 2\,HD$,

= the given differences of sides.

[Most of the solutions of 13–18 are taken from Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire* (6ème éd.), pp. 61–63.]

19. Since the two tangents that can be drawn to a circle from an external point are equal (III. 17, Cor.), there result the following equalities:

$AE = AF, AE_1 = AF_1, AE_2 = AF_2, AE_3 = AF_3;$
$BD = BF, BD_1 = BF_1, BD_2 = BF_2, BD_3 = BF_3;$
$CD = CE, CD_1 = CE_1, CD_2 = CE_2, CD_3 = CE_3.$

(1) Values of s.

$$AE_1 = AC + CE_1 = AC + CD_1,$$
and $AF_1 = AB + BF_1 = AB + BD_1;$
$$\therefore AE_1 + AF_1 = AC + AB + CD_1 + BD_1$$
$$= AC + AB + BC = 2s.$$
But $AE_1 = AF_1$; $\therefore s = AE_1 = AF_1$.
Similarly $s = BD_2 = BF_2 = CD_3 = CE_3$.

(2) Values of $s - a$, $s - b$, $s - c$.

$$AE = AF, BD = BF, CD = CE;$$
$$\therefore AE + BD + CD = AF + BF + CE.$$
But the sum of these six segments is the perimeter;
$$\therefore s = AE + BD + CD = AE + BC = AE + a.$$
Hence $s - a = AE$, and $\therefore = AF$.

Again $s = CD_3 = CB + BD_3 = a + BD_3$;
hence $s - a = BD_3$, and $\therefore = BF_3$.

Lastly $s = BD_2 = BC + CD_2 = a + CD_2$;
hence $s - a = CD_2$, and $\therefore = CE_2$.

The values for $s - b$, $s - c$ can be deduced in a manner exactly similar to the preceding.

(3) Values of a, b, c.

$$a = s - (s - a) = AE_1 - AE = EE_1;$$
$$= AF_1 - AF = FF_1;$$
$$= BF_2 - BF_3 = F_2F_3;$$
$$= CE_3 - CE_2 = E_2E_3.$$
Similarly for the values of b and c.

(4) Values of $a + b$, $b + c$, $c + a$.

$$a + b = FF_1 + FF_2 = F_1F_2.$$
$$b + c = DD_2 + DD_3 = D_2D_3.$$
$$c + a = EE_3 + EE_1 = E_1E_3.$$

(5) Values of $a \sim b$, $b \sim c$, $c \sim a$.

$$a \sim b = FF_1 \sim F_1F_3 = FF_3.$$
$$b \sim c = DD_2 \sim D_1D_2 = DD_1.$$
$$c \sim a = E_1E_2 \sim EE_1 = EE_2.$$

(6) Values of $a + b + c$.

From (1) and (2) it follows that $s + (s - a) + (s - b) + (s - c) =$ each of the six given expressions.

Now $s + (s - a) + (s - b) + (s - c)$,

$$= 4s - (a + b + c),$$
$$= 2(a + b + c) - (a + b + c),$$
$$= a + b + c.$$

(7) Values of $a^2 + b^2 + c^2$.

From (1) and (2) it follows that $s^2 + (s - a)^2 + (s - b)^2 + (s - c)^2 =$ each of the six given expressions.

Now $s^2 + (s - a)^2 + (s - b)^2 + (s - c)^2$

$$= 4s^2 - 2(a + b + c)s + a^2 + b^2 + c^2,$$
$$= 4s^2 - 4s \cdot s + a^2 + b^2 + c^2,$$
$$= a^2 + b^2 + c^2.$$

(8) $AI^2 + BI^2 + CI^2 + s^2$

$$= (AF^2 + IF^2) + (BD^2 + ID^2) + (CE^2 + IE^2) + s^2,$$
$$= AF^2 + BD^2 + CE^2 + 3r^2 + s^2,$$
$$= (s - a)^2 + (s - b)^2 + (s - c)^2 + 3r^2 + s^2,$$
$$= a^2 + b^2 + c^2 + 3r^2.$$

Similarly

$$AI_1^2 + BI_1^2 + CI_1^2 + (s - a)^2 = a^2 + b^2 + c^2 + 3r_1^2,$$
$$AI_2^2 + BI_2^2 + CI_2^2 + (s - b)^2 = a^2 + b^2 + c^2 + 3r_2^2,$$
$$AI_3^2 + BI_3^2 + CI_3^2 + (s - c)^2 = a^2 + b^2 + c^2 + 3r_3^2;$$

$\therefore\ \Sigma(AI^2) + \Sigma(BI^2) + \Sigma(CI^2) + s^2 + (s - a)^2 + (s - b)^2 + (s - c)^2$

$$= 4(a^2 + b^2 + c^2) + 3(r^2 + r_1^2 + r_2^2 + r_3^2)$$

$\therefore\ \Sigma(AI^2) + \Sigma(BI^2) + \Sigma(CI^2)$

$$= 3(a^2 + b^2 + c^2) + 3(r^2 + r_1^2 + r_2^2 + r_3^2).$$

(9) Let H be the middle point of BC.

Since BD and CD_1 are each $= s - b$, by (2)

$\therefore\ D, D_1$ are equidistant in opposite directions from B, C;

$\therefore\ D, D_1$ are equidistant from H.

Similarly D_2, D_3 are equidistant from H.

Hence also analogous relations hold for the E points and the F points.

$D_2D_3 = AB + AC,\quad DD_1 = AB \sim AC$, by (4) and (5);

$\therefore\ 2HD_2 = AB + AC,\ 2HD = AB \sim AC.$

Now $AD^2 + AD_1^2 = 2HD^2 + 2HA^2$, *App. II.* 1

$AD_2^2 + AD_3^2 = 2HD_2^2 + 2HA^2$; *App. II.* 1

$\therefore\ AD^2 + AD_1^2 + AD_2^2 + AD_3^2 = 2\,HD^2 + 2\,HD_2^2 + 4\,HA^2.$

But $\qquad 2\,AB^2 + 2\,AC^2 = 4\,HB^2 + 4\,HA^2,$

$\qquad\qquad = BC^2 + 4\,HA^2;$

$\therefore\ \quad 2\,AB^2 + 2\,AC^2 - BC^2 = 4\,HA^2.$

Hence $AD^2 + AD_1^2 + AD_2^2 + AD_3^2$

$= 2\,HD^2 + 2\,HD_2^2 + 2\,AB^2 + 2\,AC^2 - BC^2,$

$= \frac{1}{2}\left\{4\,HD^2 + 4\,HD_2^2\right\} + 2\,AB^2 + 2\,AC^2 - BC^2,$

$= \frac{1}{2}\left\{(AB+AC)^2 + (AB \sim AC)^2\right\} + 2AB^2 + 2AC^2 - BC^2,$

$= AB^2 + AC^2 + 2\,AB^2 + 2\,AC^2 - BC^2,$

$= 3\,(AB^2 + AC^2) - BC^2.$

Similarly $BE^2 + BE_1^2 + BE_2^2 + BE_3^2 = 3\,(AB^2 + BC^2) - AC^2,$

and $\qquad CF^2 + CF_1^2 + CF_2^2 + CF_3^2 = 3\,(AC^2 + BC^2) - AB^2.$

Consequently the sum of the squares on the twelve straight lines specified

$$= 3\,(AB^2 + AC^2) + 3\,(AB^2 + BC^2) + 3\,(AC^2 + BC^2) - (BC^2 + AC^2 + AB^2),$$

$$= 5\,(AB^2 + AC^2 + BC^2).$$

(10) I_1A is $\perp$ I_2I_3, since the first bisects $\angle\ BAC$, and the second the angles adjacent to BAC.

Hence also I_2B is $\perp$ I_3I_1, and I_3C is $\perp$ I_1I_2.

The right-angled $\triangle$s AIE, AI_1E_1 are mutually equiangular, since the angles at A are common.

The right-angled $\triangle$s AIE, AI_2E_2 are mutually equiangular,
since $\angle\ EAI$ is the complement of $\angle\ E_2AI_2$,
and $\angle\ AI_2E_2$ is the complement of $\angle\ E_2AI_2$.

The right-angled $\triangle$s AIE, AI_3E_3 are mutually equiangular,
since $\angle\ EAI$ is the complement of $\angle\ E_3AI_3$,
and $\angle\ AI_3E_3$ is the complement of $\angle\ E_3AI_3$.

Similarly the two other sets of four triangles may be proved mutually equiangular.

(11) AIF, AI_1F_1, AI_2F_2, AI_3F_3; BID, BI_1D_1, BI_2D_2, BI_3D_3; CIE, CI_1E_1, CI_2E_2, CI_3E_3.

(12) Because $\angle$s I_1AI_2, I_1BI_2 are right,

$\therefore$ the four points A, B, I_1, I_2 are concyclic. *III.* 21

$\triangle$s AIB, ACI_1 are mutually equiangular,

since $\angle\ BAI = \angle\ I_1AC$, *App. IV.* 1

and $\angle\ ABI = \angle\ AI_1C$. *III.* 21

$\triangle$s AIB, I_2CB are mutually equiangular,

since $\angle ABI = \angle I_2BC$, *App. IV.* 1
and $\angle BAI = \angle BI_2C$. *III.* 21
Similarly the two other sets of three triangles may be proved to be mutually equiangular from the fact that the four points B, C, I_2, I_3 are concyclic, as well as C, A, I_3, I_1.

(13) Because $\angle$s IBI_1, ICI_1 are right,
$\therefore$ the four points I, B, I_1, C are concyclic. *III.* 32
Hence by the second deduction from III. 22,
$\triangle I_1BI_2$ is equiangular to $\triangle ICI_2$,
and $\triangle I_1CI_3$ is equiangular to $\triangle IBI_3$.
Now $\triangle$s I_1BI_2, I_1CI_3 are mutually equiangular,
since they are right-angled, and have their angles at I_1 common;
$\therefore$ the first set of four triangles are mutually equiangular.
Similarly the two other sets of four triangles may be proved mutually equiangular.

(14) (*a*) Since the four points B, F_1, I_1, D_1 are concyclic,
$\therefore \angle ABC = \angle F_1I_1D_1$, *III.* 22, *Cor.*
$= 2 \angle BI_1D_1$.
Similarly $\angle ACB = 2 \angle CI_1D_1$;
$\therefore 2(\angle BI_1D_1 + \angle CI_1D_1) = \angle ABC + \angle ACB$;
$\therefore \quad \angle I_3I_1I_2 = \frac{1}{2}(B + C)$.
Similarly $\angle I_1I_2I_3 = \frac{1}{2}(C + A)$,
and $\quad \angle I_2I_3I_1 = \frac{1}{2}(A + B)$.

By the eighth deduction from III. 17, EF is $\perp AI$;
$\therefore EF$ is $\parallel I_2I_3$.
Similarly FD is $\parallel I_3I_1$, and DE is $\parallel I_1I_2$;
$\therefore \triangle DEF$ is equiangular to $\triangle I_1I_2I_3$.

$$\text{In } \triangle I_1BC, \begin{cases} \angle BI_1C = \angle I_3I_1I_2 \quad = \frac{1}{2}(B + C), \\ \angle I_1BC = \frac{1}{2} \angle F_1BC = \frac{1}{2}(C + A), \\ \angle BCI_1 = \frac{1}{2} \angle E_1CB = \frac{1}{2}(A + B). \end{cases}$$

Similarly, in $\triangle$s I_2CA, I_3AB,
$$\angle CI_2A = \frac{1}{2}(C + A),\ \angle I_2CA = \frac{1}{2}(A + B),$$
$$\angle CAI_2 = \frac{1}{2}(B + C);$$
$$\angle AI_3B = \frac{1}{2}(A + B),\ \angle I_3AB = \frac{1}{2}(B + C),$$
$$\angle ABI_3 = \frac{1}{2}(C + A).$$

Because $\angle AEF + \angle AFE =$ the supplement of A;
$\therefore \angle AEF + \angle AFE = B + C$;
$\therefore \angle AEF = \angle AFE = \frac{1}{2}(B + C)$.

Similarly $\angle\ BDF = \angle\ BFD = \frac{1}{2}(C + A)$,
and $\angle\ CDE = \angle\ CED = \frac{1}{2}(A + B)$.

(b) $\angle\ AIB$ = supplement of ($\angle\ IAB + \angle\ IBA$),
= supplement of $\frac{1}{2}(A + B)$.
Similarly $\angle\ BIC$ = supplement of $\frac{1}{2}(B + C)$,
and $\angle\ CIA$ = supplement of $\frac{1}{2}(C + A)$.

$\angle\ AI_1B$ = complement of $\angle\ AI_3I_1$,
= complement of $\frac{1}{2}(A + B) = \frac{1}{2}C$.
$\angle\ BI_1C = \frac{1}{2}(B + C)$, by (a).
$\angle\ CI_1A$ = complement of $\angle\ AI_2I_1$,
= complement of $\frac{1}{2}(C + A) = \frac{1}{2}B$.
Similarly $\angle\ AI_2B = \frac{1}{2}C$, $\angle\ BI_2C = \frac{1}{2}A$, $\angle\ CI_2A = \frac{1}{2}(C + A)$;
$\angle\ AI_3B = \frac{1}{2}(A + B)$, $\angle\ BI_3C = \frac{1}{2}A$, $\angle\ CI_3A = \frac{1}{2}B$.

(c) Since the four points C, D, I, E are concyclic, *III.* 22
$\therefore$ $\angle\ DIE$ = supplement of $C = A + B$.
Similarly $\angle\ EIF$ = supplement of $A = B + C$,
and $\angle\ FID$ = supplement of $B = C + A$.

Since the four points I_1, B, I, C are concyclic, *III.* 22
$\therefore$ $\angle\ I_1IC = \angle\ I_1BC$, *III.* 21
= complement of $\angle\ IBC$,
= complement of $\frac{1}{2}B$.
Similarly $\angle\ I_2IC$ = complement of $\frac{1}{2}A$;
$\therefore$ $\angle\ I_1II_2$ = supplement of $\frac{1}{2}(A + B)$.
Similarly $\angle\ I_2II_3$ = supplement of $\frac{1}{2}(B + C)$,
and $\angle\ I_3II_1$ = supplement of $\frac{1}{2}(C + A)$.

Since $\angle\ I_1IC$ = complement of $\frac{1}{2}B$;
and $\angle\ DIC$ = complement of $\angle\ DCI$,
= complement of $\frac{1}{2}C$;
$\therefore \angle\ I_1IC - \angle\ DIC = (1 \text{ rt. } \angle - \frac{1}{2}B) - (1 \text{ rt. } \angle - \frac{1}{2}C)$;
$\therefore$ $\angle\ I_1ID = \frac{1}{2}(C - B)$.
Similarly $\angle\ I_2IE = \frac{1}{2}(C - A)$,
and $\angle\ I_3IF = \frac{1}{2}(B - A)$.

Many of the preceding results may be obtained in other ways.

(15) These results have been established in the proof of (9).

20. It has been proved in (10) of the preceding that I_1A is $\perp$ I_2I_3, I_2B is $\perp$ I_3I_1, and I_3C is $\perp$ I_1I_2;
$\therefore$ I is the orthocentre of $\triangle\ I_1I_2I_3$, and ABC is the orthocentric triangle.

Similarly I_1 is the orthocentre of $\triangle\ II_2I_3$,
I_2 is the orthocentre of $\triangle\ II_3I_1$,
I_3 is the orthocentre of $\triangle\ II_1I_2$;
and since A, B, C are in every case the feet of the perpendiculars, ABC is the orthocentric triangle.

21. Consider $\triangle\ I_1I_2I_3$ whose orthocentre is I, and whose orthocentric triangle is ABC.
I is the inscribed centre, and the vertices I_1, I_2, I_3 are the three escribed centres of $\triangle\ ABC$.
Consider $\triangle\ II_2I_3$ whose orthocentre is I_1, and whose orthocentric triangle is ABC.
I_1 is the first escribed centre, and the vertices I, I_2, I_3 are the inscribed centre and the second and third escribed centres of $\triangle\ ABC$.
Similarly for $\triangle$s II_3I_1 and II_1I_2.

22. These six straight lines are (fig. on p. 251 of *Euclid*):
II_1, II_2, II_3, I_1I_2, I_2I_3, I_3I_1.
The circle whose diameter is II_1 passes through B, C;
the circle whose diameter is II_2 passes through C, A;
the circle whose diameter is II_3 passes through A, B;
the circle whose diameter is I_1I_2 passes through A, B;
the circle whose diameter is I_2I_3 passes through B, C;
the circle whose diameter is I_3I_1 passes through C, A.
The circle circumscribed about $\triangle\ ABC$ is the medioscribed circle of $\triangle\ I_1I_2I_3$;
$\therefore$ its $\bigcirc^{ce}$ passes through the middle points of the six specified straight lines, that is, through the centres of the six specified circles.

23. Construct the figure as in the twenty-ninth deduction following.
Then, as it is there proved, U lies on II_1, and H' on I_2I_3.
Now because ID, UH, I_1D_1 are parallel, and $DH = D_1H$,
$\therefore$ U is the middle point of II_1. *App. I.* 1, *Cor.* 2
Because I_2D_2, $H'H$, I_3D_3 are parallel, and $D_2H = D_3H$,
$\therefore$ H' is the middle point of I_2I_3. *App. I.* 1, *Cor.* 2
Similarly the circumscribed circle ABC passes through the middle points of II_2 and I_3I_1, and of II_3 and I_1I_2.
[This is another proof of the property of the medioscribed circle different from any of those to which references are given on p. 255 of *Euclid*.]

24. (1) On reference to the equalities established in (1) and (2) of the nineteenth deduction it will be seen that

$AF_3 = CD_1$, $BD_1 = AE_2$, $CE_2 = BF_3$;
$\therefore AF_3^2 + BD_1^2 + CE_2^2 = F_3B^2 + D_1C^2 + E_2A^2$;
$\therefore I_1D_1$, I_2E_2, I_3F_3 are concurrent, by the sixty-sixth deduction of Book I.

Again $AF_1 = CD_3$, $BD_3 = AE$, $CE = BF_1$;
$\therefore AF_1^2 + BD_3^2 + CE^2 = F_1B^2 + D_3C^2 + EA^2$;
$\therefore IE$, I_1F_1, I_3D_3 are concurrent.
Similarly the other two sets of three radii are concurrent.

(2) A_1I_3, S_1I_2 being both $\perp$ CA, are parallel,
and I_3S_1, I_2A_1 being both $\perp$ AB, are parallel;
$\therefore$ the figure $A_1I_3S_1I_2$ is a $\|^m$.

Again $\angle\ AI_2E_2 = \angle\ AI_2F_2$, since they are complements of the equal $\angle$ s I_2AE_2, I_2AF_2;
that is, I_2I_3, a diagonal of the $\|^m$ $A_1I_3S_1I_2$, bisects one of its $\angle$ s $A_1I_2S_1$;
$\therefore \|^m$ $A_1I_3S_1I_2$ is a rhombus.
Similarly $B_1I_1S_1I_3$ and $C_1I_2S_1I_1$ are rhombi.

Hence S_1I_1, S_1I_2, S_1I_3 are equal, and the hexagon $A_1I_3B_1I_1C_1I_2$, since its sides are equal and parallel to S_1I_1, S_1I_2, S_1I_3 is equilateral, and has its opposite sides parallel.

(3) Since A_1I_2 is equal and parallel to B_1I_1,
$\therefore A_1B_1$ is equal and parallel to I_1I_2.
Similarly B_1C_1 is equal and parallel to I_2I_3, and C_1A_1 to I_3I_1.

(4) Since S_1I_1, S_1I_2, S_1I_3 are equal, S_1 is the centre of the circle circumscribed about $\triangle\ I_1I_2I_3$.

Again the circles circumscribed about $\triangle$ s II_2I_3, II_3I_1, II_1I_2 are equal to the circle circumscribed about $\triangle\ I_1I_2I_3$, by the twentieth deduction of Book IV. and the ninth deduction of Book III.;
$\therefore$ their radii are equal to its radius, that is, $= S_1I_1$, S_1I_2, or S_1I_3.
To find the centre of the circle II_2I_3, take I_2, I_3 as centres, and S_1I_1 as radius, and describe two circles intersecting at A_1, S_1; A_1 is the centre of the circle II_2I_3. For S_1 cannot be the centre, since it is the centre of the circle $I_1I_2I_3$.
Similarly B_1, C_1 are the centres of the circles II_3I_1, II_1I_2.

(5) For A_1I, B_1I_3, being both $\perp$ BC, are parallel,
and A_1I_3, B_1I, being both $\perp$ CA, are parallel;
$\therefore$ the figure $A_1IB_1I_3$ is a $\|^m$.

Now II_3 is $\perp$ I_1I_2, by the twentieth deduction;
$\therefore$ it is $\perp$ A_1B_1, by (3),
that is, the diagonals of $A_1IB_1I_3$ intersect at right angles;
$\therefore$ $A_1IB_1I_3$ is a rhombus.
Similarly $B_1IC_1I_1$ and $C_1IA_1I_2$ are rhombi.

Hence IA_1, IB_1, IC_1 are all equal, and I is the centre of the circle circumscribed about $\triangle$ $A_1B_1C_1$.

(6) For I is the orthocentre, S_1 the circumscribed centre of $\triangle$ $I_1I_2I_3$, and A, B, C are the feet of the perpendiculars;
$\therefore$ the circle passing through A, B, C is the medioscribed circle of $\triangle$ $I_1I_2I_3$, and has its centre at the middle point of IS_1, *App. IV.* 2, *Cor.* 2
and its radius equal to half the circumscribed radius of $\triangle$ $I_1I_2I_3$. *App. IV.* 2, *Cor.* 3

Since S_1 is the orthocentre, and I the circumscribed centre of $\triangle$ $A_1B_1C_1$, the medioscribed circle of $\triangle$ $A_1B_1C_1$ has its centre at the middle point of S_1I, *App. IV.* 2, *Cor.* 2
and its radius equal to half the circumscribed radius of $\triangle$ $A_1B_1C_1$. *App. IV.* 2, *Cor.* 3
But $\triangle$s $I_1I_2I_3$ and $A_1B_1C_1$ are congruent, by (3);
$\therefore$ their circumscribed radii are equal;
$\therefore$ the medioscribed circle of $\triangle$ $A_1B_1C_1$ has the same centre and radius as the medioscribed circle of $\triangle$ $I_1I_2I_3$,
that is, the circumscribed circle of $\triangle$ ABC is the medioscribed circle of $\triangle$ $A_1B_1C_1$.
Hence the circumscribed circle of $\triangle$ ABC is the medioscribed circle of the eight triangles mentioned. *App. IV.* 2, *Cor.* 4

25. See fig. on p. 251 of *Euclid.*

$$\begin{aligned}\triangle\, ABC &= \triangle\, AIB + \triangle\, BIC + \triangle\, CIA,\\ &= \tfrac{1}{2} AB\cdot IF + \tfrac{1}{2} BC\cdot ID + \tfrac{1}{2} CA\cdot IE,\\ &= \tfrac{1}{2}\, c\cdot r + \tfrac{1}{2}\, a\cdot r + \tfrac{1}{2}\, b\cdot r,\\ &= \tfrac{1}{2}(a+b+c)\, r = rs.\end{aligned}$$

$$\begin{aligned}\triangle\, ABC &= \triangle\, AI_1B - \triangle\, BI_1C + \triangle\, CI_1A,\\ &= \tfrac{1}{2} AB\cdot I_1F_1 - \tfrac{1}{2} BC\cdot I_1D_1 + \tfrac{1}{2} CA\cdot I_1E_1,\\ &= \tfrac{1}{2}\, c\cdot r_1 - \tfrac{1}{2}\, a\cdot r_1 + \tfrac{1}{2}\, b\cdot r_1,\\ &= \tfrac{1}{2}(b+c-a)\, r_1 = r_1(s-a).\end{aligned}$$

Similarly $\triangle\, ABC = r_2(s-b) = r_3(s-c)$.

26. It was proved in the twenty-second deduction that the circle whose diameter is II_1 (fig. on p. 251 of *Euclid*) passes through B and C, and that its centre was situated on the $\bigcirc^{ce}$ of the circle circumscribed about $\triangle$ ABC.

27. Let $\triangle\ ABC$ be right-angled at C (fig. on p. 251 of *Euclid*).
Then $IDCE$ is a square; $\therefore\ ID + IE = CD + CE$;
$\therefore\ AB + ID + IE = AF + BF + CD + CE,$
$= AE + BD + CD + CE,$ *III.* 17, *Cor.*
$= AC + BC.$

28. Let $\triangle\ ABC$ be right-angled at A; for convenience let AB be greater than AC, and let the inscribed circle touch the hypotenuse at D: to prove $BD \cdot DC = \triangle\ ABC$.

Bisect BC at H.

Then $HD = \frac{1}{2}(AB - AC)$, by (5) of the nineteenth deduction,

and $BD \cdot DC = (BH + HD) \cdot (BH - HD),$
$= BH^2 - HD^2,$ *II.* 5. *Cor.*
$= \frac{1}{4} BC^2 - \left\{\frac{1}{2}(AB - AC)\right\}^2,$
$= \frac{1}{4}(AB^2 + AC^2) - \frac{1}{4}(AB^2 - 2\,AB \cdot AC + AC^2),$ *I.* 47, *II.* 7
$= \frac{1}{2} AB \cdot AC,$
$= \triangle\ ABC.$

29. Let ABC be a triangle circumscribed by a circle whose centre is S; I, I_1, I_2, I_3 the inscribed and escribed centres; ID, I_1D_1, I_2D_2, I_3D_3 the inscribed and escribed radii drawn to BC. From S draw $SH \perp BC$, and let SH meet the circumscribed circle below the base BC at U, and above it at H'.

Because SH is $\perp BC$, H is the middle point of BC, *III*, 3 and U the middle point of arc BUC.

But since AI_1 bisects $\angle\ BAC$,
$\therefore$ it bisects arc BUC; *III.* 26
that is, AI_1 passes through U.
Since UH' is a diameter, and $\angle\ UAI_3$ is right,
$\therefore\ I_2I_3$ passes through H'.
Now because ID, UH, I_1D_1 are parallel, and $DH = D_1H$,
$\therefore\ 2\ UH = I_1D_1 - ID$; *App. I.* 1, *Cor.* 2
and because I_2D_2, $H'H$, I_3D_3 are parallel, and $D_2H = D_3H$,
$\therefore\ 2\ H'H = I_2D_2 + I_3D_3$ *App. I.* 1, *Cor.* 2
Hence $2\,(UH + H'H) = I_1D_1 + I_2D_2 + I_3D_3 - ID$;
$\therefore\ 2\ H'U$ or $4R \quad = r_1 + r_2 + r_3 - r.$

30. In the figure to the preceding deduction, let SK, which is $\perp CA$, meet the circumscribed circle at V and K', and SL, which is $\perp AB$, meet the circumscribed circle at W and L'.

Then $HU = H'U - H'H = 2R - \frac{1}{2}(r_2 + r_3)$, by the preceding deduction,

$$KV = K'V - K'K = 2R - \tfrac{1}{2}(r_3 + r_1),$$
$$LW = L'W - L'L = 2R - \tfrac{1}{2}(r_1 + r_2);$$
$$\therefore\ HU + KV + LW = 6R - (r_1 + r_2 + r_3),$$
$$= 6R - (4R + r),$$
$$= 2R - r.$$

Now $SH + SK + SL = 3R - (HU + KV + LW)$,
$$= 3R - (2R - r),$$
$$= R + r.$$

Again, if O be the orthocentre of $\triangle\ ABC$,

$SH + SK + SL = \frac{1}{2}(OA + OB + OC)$; *App. I. 5, Cor.*

$\therefore\ OA + OB + OC = 2(R + r)$.

31. The circumscribed centre and the orthocentre of a triangle are outside the triangle when it is obtuse-angled. If $\angle\ A$, for example, be obtuse, SH and OA are considered negative, and the two results of the preceding deduction are then

$$-SH + SK + SL = R + r,$$

and $\quad -OA + OB + OC = 2(R + r)$.

32. By (14) (a) of the nineteenth deduction,

$A_1 = \frac{1}{2}(B + C)$, $B_1 = \frac{1}{2}(C + A)$, $C_1 = \frac{1}{2}(A + B)$;

$\therefore\ B_1 - A_1 = \frac{1}{2}(A - B)$, $C_1 - B_1 = \frac{1}{2}(B - C)$,

$C_1 - A_1 = \frac{1}{2}(A - C)$.

Similarly $A_2 = \frac{1}{2}(B_1 + C_1)$, $B_2 = \frac{1}{2}(C_1 + A_1)$, $C_2 = \frac{1}{2}(A_1 + B_1)$;

$\therefore\ A_2 - B_2 = \frac{1}{2}(B_1 - A_1) \quad = \frac{1}{4}(A - B)$,

$B_2 - C_2 = \frac{1}{2}(C_1 - B_1) \quad = \frac{1}{4}(B - C)$,

$A_2 - C_2 = \frac{1}{2}(C_1 - A_1) \quad = \frac{1}{4}(A - C)$, and so on.

Hence the differences between the angles of the successive triangles become always a smaller and smaller fraction of the differences between the angles of the original triangle; the successive triangles therefore approximate to an equilateral triangle.

33. Let $ABCDE$ (fig. to IV. 13) be an equilateral polygon circumscribed about a circle whose centre is O.

Join OB, OC, OD, OF, OG.

Then $\triangle$s OFC, OGC are congruent; *I. 4 or 8*

$\therefore\ OC$ bisects $\angle\ BCD$.

Similarly OB, OD bisect $\angle$s ABC, CDE.

Because $\triangle$s OCB, OCD are congruent, *I. 4*

$\therefore\ \angle\ OBC = \angle\ ODC$;

$\therefore\ \angle\ ABC = \angle\ CDE$.

Similarly $\angle\ CDE = \angle\ EAB$, $\angle\ EAB = \angle\ BCD$, $\angle\ BCD = \angle\ DEA$;

$\therefore$ all the angles are equal.

This course of reasoning applied to a five-sided figure will apply equally well to a figure with any odd number of sides; for it shows that those angles are equal which alternate with one another going cyclically round the polygon.

If the number of sides of the circumscribed equilateral polygon be even, the course of reasoning shows only that the 1st, 3rd, 5th, &c. angles are equal, and that the 2nd, 4th, 6th, &c. angles are equal.

34. From O draw $OF \perp AB$, and $OG \perp CD$.

Then $\triangle$s AOF, COG are congruent, and $\angle\ AOF = \angle\ COG$;

$\therefore\ \angle\ AOC = \angle\ FOG$.

Now $\angle\ FOG$ is supplementary to $\angle\ E$, since the points O, F, E, G are concyclic; *III.* 22

$\therefore\ \angle\ AOC$ is supplementary to $\angle\ E$;

$\therefore$ the points A, E, C, O are concyclic. *III.* 22

Similarly the points D, E, B, O are concyclic.

35. Let $ABCDE$ (fig. to IV. 13) be a regular n-gon.

Then $\triangle\ OCD = \frac{1}{2}\ CD \cdot OG$;

$\therefore$ area of the n-gon $= n \times \frac{1}{2}\ CD \cdot OG$,

$= n \times \frac{1}{2}$ (any side) $\cdot$ (inscribed radius).

Now if from P, any point inside the n-gon, perpendiculars be drawn to the sides, and if P be joined with the vertices of the n-gon, n triangles will be formed, the sum of whose areas $= \frac{1}{2}$ (any side) $\cdot$ (sum of perpendiculars).

But the sum of the areas of these n triangles = the area of the n-gon;

$\therefore\ n \times \frac{1}{2}$ (any side) $\cdot$ (inscribed radius) $= \frac{1}{2}$ (any side) $\cdot$ (sum of perpendiculars);

$\therefore\ n \times$ inscribed radius = sum of perpendiculars.

When the point P is outside the n-gon, one or more of the perpendiculars would be considered negative, according to the position of P; and the sum of the perpendiculars would then be their algebraic sum. For example, if P were situated between BC and ED produced, the perpendicular on CD would be considered negative, the other perpendiculars positive; if P were situated between CD and ED produced, the perpendiculars on CD and ED produced would be considered negative, the other perpendiculars positive.

LOCI.

1. Let ABC (fig. on p. 253 of *Euclid*) be one of the triangles on the given base BC, and having its vertical $\angle A$ = the given vertical angle; and let O be its orthocentre.

Because the points A, Z, O, Y are concyclic,

$\therefore \angle ZOY$ = the supplement of $\angle A$; *III.* 22

$\therefore \angle BOC$ = the supplement of $\angle A$; *I.* 15

$\therefore$ the locus of O is the arc of a segment of a circle described on BC, and containing an angle supplementary to the given vertical angle.

2. Let ABC (fig. on p. 251 of *Euclid*) be one of the triangles on the given base BC, and having its vertical $\angle A$ = the given vertical angle.

By (14) (b) of the nineteenth deduction,

$\angle BIC$ = supplement of $\frac{1}{2}(B + C)$.

Now since $\angle A$ is given, $\angle B + \angle C$ is given; *I.* 32

$\therefore \frac{1}{2}(B + C)$ is given;

$\therefore$ the supplement of $\frac{1}{2}(B + C)$ is given;

$\therefore$ the locus of I is the arc of a segment of a circle described on BC, and containing an angle supplementary to $\frac{1}{2}(B + C)$.

3. Since $\angle BI_1C = \frac{1}{2}(B + C)$, by (14) ($b$) of the nineteenth deduction;

$\therefore$ the locus of I_1 is the arc of a segment of a circle described on BC, and containing an angle equal to $\frac{1}{2}(B + C)$.

Since $\angle BI_2C$ and $\angle BI_3C$ are each $= \frac{1}{2}A$, by (14) (b) of the nineteenth deduction;

$\therefore$ the locus of I_2 and I_3 is the arc of a segment of a circle described on BC, and containing an angle equal to half the given vertical angle.

4. Let G (fig. on p. 100 of *Euclid*) be the centroid of $\triangle ABC$.

Through G draw $GD \parallel AB$, and $GE \parallel AC$, meeting BC at D and E.

Since $BG = 2\,GK$, $\therefore BE = 2\,EC$.

Similarly $CD = 2\,DB$;

$\therefore D$ and E are the points of trisection of BC;

$\therefore DE$ is fixed.

Now $\angle DGE = \angle BAC$; *I.* 34, *Cor.*

$\therefore$ the locus of G is the arc of a segment of a circle described

on DE, and containing an angle equal to the given vertical angle.

5. By the thirtieth deduction of Book III., the locus of the other point of intersection will be part of the $\bigcirc^{ce}$ of the circle circumscribed about $\triangle ABC$.

6. Let D be in AB, E in AC.

Since DE is of given length, and $\angle DAE$ is fixed,

$\therefore$ the circle circumscribed about $\triangle ADE$ is constant in magnitude.

Now the circle circumscribed about $\triangle ADE$ will pass through P, since $\angle$s ADP, AEP are right; *III.* 21

and of that circle AP will be a diameter;

$\therefore$ the length of AP is constant,

$\therefore$ the locus of P is a circle whose centre is A.

Let DO which is $\perp AC$, and EO which is $\perp AB$, meet AC and AB respectively at F and G.

Then AO is a diameter of the circle which passes through A, F, G, because $\angle$s AFO, AGO are right;

$\therefore$ the length of AO will be constant, and the locus of O, a circle whose centre is A, if it can be proved that the circle which passes through A, F, G is constant in magnitude.

Now the points E, F, D, G lie on the circle of which DE is a diameter; and since DE is of given length, this circle is constant in magnitude.

Again $\angle FEG$ in the segment FEG is a constant angle, for it is complementary to the given angle;

$\therefore$ the chord FG is constant.

Hence, since FG is constant, and $\angle FAG$ constant, the circle which passes through A, F, G is constant in magnitude.

7. Let ABC be one of the triangles having the given vertical $\angle A$, and the sum of the sides AB, AC given.

About $\triangle ABC$ circumscribe a circle; let P be the middle of arc BC, and join AP. From P draw $PS \perp AB$.

Then AP bisects $\angle BAC$; *III.* 27

and $AS = \frac{1}{2}(AB + AC)$, by the twenty-fifth deduction of Book IV.,

$= $ a fixed length.

Hence $\triangle ASP$ could be constructed from what is known, by the tenth deduction from I. 23;

in other words, AP is a constant length.

All the circles, therefore, circumscribed about the triangles

fulfilling the given conditions, pass through the two fixed points A and P;

$\therefore$ the locus of their centres is the perpendicular to AP at its middle point.

8. Let OX, OY be the two fixed straight lines, and let D be the centre of the circle in which a $\triangle\ ABC$ is inscribed, having $AB \parallel OX$ and $AC \parallel OY$.

First suppose AB, AC to be drawn in the same directions from A, as OX, OY are from O.

From D draw perpendiculars to OX, OY, meeting the arcs AB, AC at E, F; join BF, CE, intersecting at I.

The perpendiculars to OX, OY are $\perp\ AB$, AC;

$\therefore$ E, F are the middle points of the arcs AB, AC; *III.* 30

$\therefore$ BF, CE are the bisectors of $\angle$s ABC, ACB; *III.* 27

$\therefore$ I is the inscribed centre of $\triangle\ ABC$. *IV.* 4

If $\triangle\ ABC$ vary in such a manner that, while AB and AC always remain $\parallel$ OX and OY, AB diminishes indefinitely, the point I, where BF and CE meet, will approach nearer and nearer to E, and ultimately, when the direction of AB is that of the tangent at E, and $\triangle\ ABC$ has become infinitely thin, coincide with it;

$\therefore$ E is a point on the locus.

Similarly F is a point on the locus; and E and F are fixed points.

Now since $\angle\ EIF$ = supplement of $\frac{1}{2}(B + C)$, by (14) (*b*) of the nineteenth deduction of Book IV.,

= a right angle + $\frac{1}{2}\ \angle\ A$,

= a right angle + $\frac{1}{2}\ \angle\ XOY$;

$\therefore$ the locus of I is the arc of a segment of a circle described on the chord EF, and containing an angle = a right angle + $\frac{1}{2}\ \angle\ XOY$.

Next, let ED, FD be produced to meet the $\bigcirc^{ce}$ again at E', F'. When AB, AC are drawn from A in directions opposite to those in which OX, OY are drawn from O, the locus of the inscribed centre of $\triangle\ ABC$ is the arc of a segment of a circle described on the chord $E'F'$, and containing an angle = a right angle + $\frac{1}{2}\ \angle\ XOY$.

When AB is drawn in the same direction as OX, and AC in the direction opposite to OY, the locus of the inscribed centre of $\triangle\ ABC$ is the arc of a segment of a circle described

on the chord $E'F$, and containing an angle supplementary to $\frac{1}{2} \angle XOY$.

When AB is drawn in the direction opposite to OX, and AC in the same direction as OY, the locus of the inscribed centre of $\triangle ABC$ is the arc of a segment of a circle described on the chord EF', and containing an angle supplementary to $\frac{1}{2} \angle XOY$.

9. Let O be the centre of the two concentric circles. Join OP, OQ, OR; let OP meet the inner circle at S, and join QS.

Then $\angle PQS = \frac{1}{2} \angle QOS$, *III.* 32, 20

and $\angle PQR = \frac{1}{2} \angle QOR$. *III.* 32, 20

But $\angle QOR = 2 \angle QOS$;

$\therefore \angle PQR = 2 \angle PQS$;

$\therefore \angle PQR$ is bisected by QS.

Similarly $\angle PRQ$ is bisected by RS;

$\therefore S$ is the inscribed centre of $\triangle PQR$; *IV.* 4

$\therefore$ the locus of the inscribed centre of $\triangle PQR$ is the $\bigcirc^{ce}$ of the inner circle.

Again, since $\angle$s OQP, ORP are right, *III.* 18

$\therefore$ the points O, Q, P, R are concyclic; *III.* 22

$\therefore$ the circumscribed circle of $\triangle PQR$ always passes through O, and its centre is the middle point of OP;

$\therefore$ the locus of the circumscribed centre of $\triangle PQR$ is the $\bigcirc^{ce}$ of a circle whose centre is O, and radius half the radius of the outer circle.

10. Let $ABCD$ be the given square, and P a point outside it such that PB, PC trisect $\angle APD$.

From B draw $BE \perp PA$ produced, and $BF \perp PC$.

Since BP bisects $\angle EPF$,

$\therefore \triangle$s BEP, BFP are congruent, and $BE = BF$; *I.* 26

$\therefore \triangle$s BAE, BCF are congruent, and $\angle BAE = \angle BCF$.
I. A, Cor.

Now $\angle BAP$ is supplementary to $\angle BAE$; *I.* 13

$\therefore \angle BAP$ is supplementary to $\angle BCP$;

$\therefore$ the points A, B, C, P are concyclic; *III.* 22

$\therefore$ the locus of P is the $\bigcirc^{ce}$ of the circle which passes through A, B, C, that is, the $\bigcirc^{ce}$ of the circle circumscribed about the square.

BOOK VI.

PROPOSITION 1.

1. These two theorems are proved by the method of *reductio ad absurdum.*
2. Let A and B be the two straight lines.
   Then $A^2 : A \cdot B = A : B$, and $A \cdot B : B^2 = A : B$; *VI.* 1
   $\therefore \quad A^2 : A \cdot B = A \cdot B : B^2$.
3. For $A \cdot C : B \cdot C = A : B$. *VI.* 1
4. The triangles into which the diagonal divides the quadrilateral, being on the same base, are as their altitudes, and are therefore equal.
   Hence (I. 38) the straight lines drawn from the middle point of the diagonal to the opposite vertices, divide the quadrilateral into four equal triangles.
5. Join AC, BD.
   Then $\triangle AEC = \triangle BED$, by the second deduction from I. 37;
   $\therefore \quad \triangle AEB : \triangle AEC = \triangle AEB : \triangle BED$.
   Now $\triangle AEB : \triangle AEC = BE : EC$,
   and $\triangle AEB : \triangle BED = AE : ED$; *VI.* 1
   $\therefore BE : EC = AE : ED$.
6. If $\triangle ABC$ be applied to $\triangle DEF$, so that A may fall on D and AB along DE, then B will fall on E, because $AB = DE$; and AC will fall along DF, because $\angle A = \angle D$.
   Hence the altitudes of $\triangle$s ABC, DEF measured from B and E are equal;
   $\therefore \triangle ABC : \triangle DEF = AC : DF$. *VI.* 1
7. Because $\triangle ABD : \triangle ADC = BD : DC$,
   and $\triangle OBD : \triangle ODC = BD : DC$;
   $\therefore \triangle ABD - \triangle OBD : \triangle ADC - \triangle ODC = BD : DC$;
   $\therefore \triangle AOB : \triangle AOC = BD : DC$.
   Similarly for the other proportions.

8. Join DF.

$$\begin{array}{lll}
\text{Then} & \triangle BAE : \triangle BED = AE : ED, & \\
\text{and} & \triangle FAE : \triangle FED = AE : ED; & VI.\ 1 \\
\therefore & \triangle BAE = \triangle BED, \text{ and } \triangle FAE = \triangle FED; & \\
\therefore & \triangle BAF = \triangle BDF. & \\
\text{Now} & \triangle BDF : \triangle BCF = BD : BC, & VI.\ 1 \\
& \qquad = 1 : 2; & \\
\therefore & \triangle BDF = \tfrac{1}{2} \triangle BCF; & \\
\therefore & \triangle BAF = \tfrac{1}{2} \triangle BCF. & \\
\text{Now} & \triangle BAF : \triangle BCF = AF : CF; & VI.\ 1 \\
\therefore & CF = 2\,AF. &
\end{array}$$

9. Let CO produced meet AB at F.

$$\begin{array}{lll}
\text{Then} & \triangle ABE : \triangle CBE = AE : EC, & VI.\ 1 \\
& \qquad = 3 : 1; & \\
\text{and} & \triangle AOE : \triangle COE = 3 : 1; & \\
\therefore & \triangle ABO : \triangle CBO = 3 : 1. & \\
\text{Now} & \triangle CBO : \triangle DBO = CB : BD, & VI.\ 1 \\
& \qquad = 4 : 1; & \\
\therefore & \triangle ABO : \triangle DBO = 12 : 1; & \\
\therefore & AO : DO = 12 : 1; & \\
\therefore & \triangle AOC : \triangle DOC = 12 : 1; & \\
\therefore & \triangle AOC : \triangle BOC = 9 : 1, & \\
\text{since} & \triangle DOC = \tfrac{3}{4} \triangle BOC. & \\
\text{But} & \triangle AOC : \triangle AOF = CO : OF, & VI.\ 1 \\
& \qquad = \triangle BOC : \triangle BOF; & VI.\ 1 \\
\therefore & \triangle AOC : \triangle BOC = \triangle AOF : \triangle BOF, \text{ by alternation}; & \\
\therefore & 9 : 1 = \triangle AOF : \triangle BOF, & \\
& \qquad = AF : FB. & VI.\ 1
\end{array}$$

10. Let P be the point within the given equilateral triangle ABC; join PA, PB, PC.

Then $\triangle$s PAB, PBC, PCA, ABC are on equal bases, and hence are to one another as their altitudes, that is, as the perpendiculars from P on the sides, and the perpendicular from A on BC.

But the sum of $\triangle$s PAB, PBC, $PCA = \triangle ABC$;

$\therefore$ the sum of the perpendiculars from P on the sides $=$ the perpendicular from A on BC, which is invariable.

11. Let A, A' be two triangles or $\|^{ms}$, and let their respective altitudes be a, a', and bases b, b'.

Take another triangle or $\|^{m}$ C, whose altitude is a, and base b'.

Then $A : C = b : b'$, *VI.* 1, *Cor.* 1
and $C : A' = a : a'$; *VI.* 1, *Cor.* 2
$\therefore \quad A : A' = \left\{ \begin{matrix} b : b' \\ a : a' \end{matrix} \right\}.$

PROPOSITION 2.

1. Since $AL = LB$ (fig. to App. I. 1), and $AK = KC$,
$\therefore\ AL : LB = AK : KC$;
$\therefore\ LK$ is $\parallel BC$. *VI.* 2
2. Let LK, drawn from L, the middle point of AB, be $\parallel BC$,
Then $AL : LB = AK : KC$. *VI.* 2
Now $AL = LB$; $\therefore\ AK = KC$.
3. Let the two straight lines ABC, DEF be cut by three parallels at the points A, B, C, and D, E, F respectively: to prove $AB : BC = DE : EF$.
Join AF, cutting BE at G.
Then $AB : BC = AG : GF$, *VI.* 2
$= DE : EF$. *VI.* 2
4. This is the previous deduction in other words.
5. Because $BD : DA = CE : EA$ (fig. 1 to VI. 2), *VI.* 2
$\therefore\ BD + DA : DA = CE + EA : EA$, by addition; *V.* 18
$\therefore\ BA : AD = CA : AE$.
Conversely, since $BA : AD = CA : AE$,
$\therefore\ BA - AD : AD = CA - AE : AE$, by subtraction; *V.* 17
$\therefore\ BD : AD = CE : AE$;
$\therefore\ DE$ is $\parallel BC$. *VI.* 2
The preceding proof is easily adapted to figs. 2 and 3 to VI. 2; the second line of the proof will be either
$\therefore\ DA - BD : DA = EA - CE : EA$, by subtraction, or
$\therefore\ BD - DA : DA = CE - EA : EA$, by subtraction.
6. Through P draw $PD \parallel AB$; from BC cut off $DE = BD$; join EP, and produce it to meet AB at F.
Then $BD : DE = FP : PE$. *VI.* 2
Now $BD = DE$; $\therefore\ FP = PE$.
7. For $\triangle FBD : \triangle AFE = BF : FA$, *VI.* 1
$= BD : DC$, *VI.* 2
$= AE : EC$, *VI.* 2
$= \triangle AFE : \triangle EDC$. *VI.* 1

The same proof is applicable to the case when D is in BC produced.

8. For $AF : FC = BE : EC$, because EF is $\parallel BA$;
$= DG : GC$, because EG is $\parallel BD$;
$\therefore$ FG is $\parallel AD$.

The same proof is applicable to the case when E is in BC produced.

9. Let AF meet BC at G.
Because $\triangle CBD = \triangle BCE$, *I.* 37
$\therefore$ $\triangle FBD = \triangle FEC$.
Now $BD : DA = CE : EA$, *VI.* 2
and $BD : DA = \triangle FBD : \triangle FDA$, *VI.* 1
and $CE : EA = \triangle FEC : \triangle FEA$; *VI.* 1
$\therefore$ $\triangle FBD : \triangle FDA = \triangle FEC : \triangle FEA$.
But $\triangle FBD = \triangle FEC$;
$\therefore$ $\triangle FDA = \triangle FEA$.
Since $\triangle FDA = \triangle FEA$, and $\triangle FDB = \triangle FEC$,
$\therefore$ $\triangle AFB = \triangle AFC$.
Now $\triangle AFB : \triangle BFG = AF : FG$, *VI.* 1
$= \triangle AFC : \triangle CFG$; *VI.* 1
$\therefore$ $\triangle BFG = \triangle CFG$, and $BG = CG$.

When D and E are on AB, AC produced either below the base or through the vertex, the preceding proof will apply word for word.

10. Because EC and DF are equal and parallel,
$\therefore$ ED is $\parallel CF$; *I.* 33
$\therefore$ $AE : EC = AG : GH$, *VI.* 2
and $BF : FD = BH : HG$. *VI.* 2
Now $AE = EC$, and $BF = FD$;
$\therefore$ AG and BH are each $= GH$.

11. Draw a straight line AE (fig. to VI. 10) making any angle with AB, and take any point F in it.
Join CF, and through B draw $BG \parallel CF$.
Cut off $FH = FG$; join HB, and through F draw $FD \parallel HB$, meeting AB produced at D.
Then $AD : DB = AF : FH$, *VI.* 2
$= AF : FG$,
$= AC : CB$. *VI.* 2

PROPOSITION 3.

1. Because CD bisects $\angle\ ACB$, $\therefore AC : CB = AD : DB$.
   Now $AC = CB$; $\therefore AD = DB$.
2. Let CD (fig. to I. 10) bisect $\angle\ ACB$ and AB.
   Because CD bisects $\angle\ ACB$, $\therefore AC : CB = AD : DB$.
   Now $AD = DB$; $\therefore AC = CB$.
3. For (fig. to VI. 3) $BA : AC = BD : DC$,
   $= \triangle\ ABD : \triangle\ ADC$. *VI.* 1
4. For $AD : DB = AE : EB$, and $AD : DC = AF : FC$.
   But $AD : DB = AD : DC$; $\therefore AE : EB = AF : FC$;
   $\therefore EF$ is $\parallel BC$. *VI.* 2
5. Let BC (fig. to VI. 3) be the straight line to be trisected.
   On BC construct a triangle ABC having BA = twice AC, and bisect $\angle\ BAC$ by AD.
   Then $BD : DC = BA : AC = 2 : 1$;
   $\therefore BC$ = thrice DC, or D is a point of trisection of BC.
6. On the given straight line as base construct a triangle whose sides shall be to one another as 3 to 2, and bisect the vertical angle.
7. On the given straight line as base construct a triangle whose sides shall be to one another as $n - 1$ to 1, and bisect the vertical angle.
8. Let ABC be a triangle, and let $\angle$s B and C be bisected by BI and CI which meet at I. Join AI, and produce it to meet BC at N.
   Then $AB : BN = AI : IN$, because BI bisects $\angle\ B$;
   $= AC : CN$, because CI bisects $\angle\ C$;
   $\therefore AB : AC = BN : CN$, by alternation;
   $\therefore AN$ bisects $\angle\ A$.
9. $BD : DC = BA : AC = c : b$;
   $\therefore BD : BD + DC = c : c + b$;
   $\therefore BD : a = c : c + b$;
   $\therefore BD = \dfrac{ac}{c+b}$. Hence $DC = BC - BD = \dfrac{ab}{c+b}$.
10. Because AB is a diameter, and CD is $\perp AB$,
    $\therefore A$ is the middle point of the arc CAD, and B the middle point of the arc CBD;
    $\therefore \angle\ CFE = \angle\ DFE$, and $\angle\ CGE = \angle\ DGE$;
    $\therefore CF : FD = CE : ED$, and $CG : GD = CE : ED$.

11. Because $BA : AC = BD : DC$,

$$\therefore\ BA + AC : BA - AC = BD + DC : BD - DC,$$
$$= \tfrac{1}{2}(BD + DC) : \tfrac{1}{2}(BD - DC),$$
$$= HC : HD.$$

12. By the second deduction, if a straight line bisect both the base and the vertical angle of a triangle, the triangle must be isosceles, and the bisector will be perpendicular to the base. Hence if the two trisectors of an angle of a triangle trisected the opposite side, both trisectors would be perpendicular to that side; which is impossible.

PROPOSITION A.

1. The more and more nearly equal AB and AC become, the more and more nearly parallel do AD and BC become, and consequently the point D is removed farther and farther from B and C. When $AB = AC$, AD is then $\parallel BC$, and BD, CD, being both infinitely long, may be said to have their ratio = the ratio of AB to AC.

2. Let ABC be a triangle, and let the exterior angles B and C be bisected by BI_1, and CI_1, which meet at I_1. Join AI_1, and let it meet BC at N.

   Then $AB : BN = AI_1 : I_1N$, because BI_1 bisects exterior $\angle B$;
   $= AC : CN$, because CI_1 bisects exterior $\angle C$;
   $\therefore\ AB : AC = BN : CN$, by alternation;
   $\therefore\ AN$ bisects $\angle A$.

3. $BD : DC = BA : AC = c : b$;
   $\therefore\ BD : BD - DC = c : c - b$;
   $\therefore\ BD : a = c : c - b$;
   $\therefore\ BD = \dfrac{ac}{c-b}$. Hence $DC = BD - BC = \dfrac{ab}{c-b}$.

4. Because AB is a diameter, and CD is $\perp AB$,
   $\therefore\ A$ is the middle point of the arc CAD, and B the middle point of the arc CBD;
   $\therefore\ AG$ would bisect $\angle CGD$, and BF would bisect $\angle CFD$.
   Now BE is $\perp AG$, and $AE \perp BF$; *III.* 31
   $\therefore\ BE$ bisects the exterior vertical angle of $\triangle CGD$, and AE bisects the exterior vertical angle of $\triangle CFD$.
   Hence $CG : GD = CE : ED$, and $CF : FD = CE : ED$.

5. Join AP, BP, which are $\perp$ each other. *III.* 31

It may be proved that in $\triangle$ PCD, PB bisects the vertical $\angle$ DPC, and PA bisects the exterior vertical angle;

$\therefore$ $DB : BC = DP : PC$, *VI.* 3

$= DA : AC$; *VI.* A

$\therefore$ $DB : DA = BC : AC$, by alternation;

$\therefore$ $AD : DB = AC : CB$, by inversion.

6. On AB as diameter describe a circle AEB; bisect the arc AEB at E; join EC, and produce it to meet the circle at F; from F draw $FD \perp FE$, and meeting AB at D. Join AF, BF.

Since arc AE = arc EB, $\therefore$ FC bisects $\angle$ AFB;

$\therefore$ $AC : CB = AF : FB$. And since FD is $\perp$ FC,

$\therefore$ FD bisects the exterior vertical angle of $\triangle$ FAB;

$\therefore$ $AD : DB = AF : FB$. Hence $AD : DB = AC : CB$.

7. From BA produced cut off $AE = AC$, and join DE.

Then $\triangle$s CAD, EAD are congruent; *I.* 4

$\therefore$ $DC = DE$, and $\angle$ CDE is bisected by DA;

$\therefore$ $BD : DE = BA : AE$; *VI.* 3

$\therefore$ $BD : DC = BA : AC$.

8. Let XY and BA be produced to meet at D, and produce XZ to E.

Because $\angle$ $AZY = \angle$ BZX, *Hyp.*

$= \angle$ AZE; *I.* 15

$\therefore$ $XZ : ZY = XD : DY$; *VI.* A

$\therefore$ $XZ : XD = ZY : DY$, by alternation.

Because $\angle$ $AYZ = \angle$ CYX, *Hyp.*

$= \angle$ AYD; *I.* 15

$\therefore$ $ZY : DY = ZA : AD$; *VI.* 3

$\therefore$ $ZX : XD = ZA : AD$;

$\therefore$ AX bisects $\angle$ ZXY. *VI.* 3

Now $\angle$ $BXZ = \angle$ CXY;

$\therefore$ $\angle$ $AXB = \angle$ AXC, or AX is $\perp$ BC.

Similarly BY is $\perp$ CA, and CZ is $\perp$ AB.

When X and Y are on BC and AC produced, $\angle$ ACB is obtuse, and the preceding proof is applicable with some slight modifications, the principal ones being that, instead of the vertical angle of a triangle being bisected, it is the exterior vertical angle, or *vice versâ*, and that the authorities VI. 3 and VI. A have to be interchanged.

PROPOSITION 4.

1. Let DE (fig. 1 to VI. 2) be drawn $\parallel BC$, the base of $\triangle ABC$.
   Then $\angle ADE = \angle ABC$, and $\angle AED = \angle ACB$;
   $\therefore$ $\triangle$s ADE, ABC are mutually equiangular;
   $\therefore$ they are similar.
2. They are mutually equiangular (I. 32, Cor. 1), and therefore similar.
3. They are mutually equiangular, by the second deduction from I. 32, and therefore similar.
4. A rhombus is a $\parallel^m$, by the fifth deduction from I. 34;
   $\therefore$ $\angle ADE = \angle CFD$, and $\angle AED = \angle CDF$; *I.* 29
   $\therefore$ $\triangle$s EAD, DCF are mutually equiangular.
5. For $\angle BAE = \angle CDE$ (III. 21), and $\angle AEB = \angle DEC$;
   $\therefore$ $\triangle$s AEB, CED are mutually equiangular.
   Again $\angle BEC = \angle DEA$, $\angle EBC = \angle EAD$;
   III. 21, or *III.* 22, *Cor.*
   $\therefore$ $\triangle$s BEC, AED are mutually equiangular.
6. By the first deduction from VI. 2, LK (fig. to App. I. 1) is $\parallel BC$;
   $\therefore$ $\triangle$s ALK, ABC are similar; $\therefore$ $AL : AB = LK : BC$.
   Now AL is half of AB; $\therefore$ LK is half of BC.
7. Let LK (fig. to App. I. 1) be $\parallel BC$ and = half of BC.
   Then $\triangle$s ALK, ABC are similar, by the first deduction;
   $\therefore$ $AL : AB = LK : BC$.
   Now LK is half of BC; $\therefore$ AL is half of AB.
   Similarly AK is half of AC.
8. Let $ABCD$ be a trapezium, having CD = twice AB, and let the diagonals AC, BD intersect at E.
   Then $\triangle$s AEB, CED are mutually equiangular;
   $\therefore$ $AB : CD = AE : CE$.
   Now $CD = 2\,AB$;
   $\therefore$ $CE = 2\,AE$; $\therefore$ $AC = 3\,AE$.
9. Let $\triangle$s ABC, DEF be mutually equiangular, and from A and D let AG, DH be drawn respectively $\perp BC$ and EF.
   Then $\triangle$s ABG, DEH are similar, by the second deduction;
   $\therefore$ $AG : DH = AB : DE$,
   $= BC : EF$.
10. Let ABC be a triangle, AH the median to the base BC, and DE a parallel to BC cutting the median at F.

Then $\triangle$s ADF, ABH are similar; $\therefore AF : AH = DF : BH$.
And $\triangle$s AEF, ACH are similar; $\therefore AF : AH = EF : CH$;
$\therefore DF : BH = EF : CH$.
Now $BH = CH$; $\therefore DF = EF$.

11. Because $\triangle$s AEF, ABC are similar, $\therefore AF : AC = EF : BC$.
Because $\triangle$s AGF, ADC are similar, $\therefore AF : AC = GF : DC$;
$\therefore EF : BC = GF : DC$;
$\therefore EF : FG = BC : CD$ by alternation.

12. Let BD be the given straight line, and let it be required to bisect it.
Draw any straight line $EFG \parallel BD$, and make $EF = FG$; join BE, DG and let them meet at A; join AF and let it meet BD at C.
Then by the preceding deduction $EF : FG = BC : CD$.
But $EF = FG$; $\therefore BC = CD$.
The method is similar for any other number of equal parts than two.

13. Apply $\triangle\ DCE$ to $\triangle\ ABC$ so that D falls on A, and DC along AB; then DE will fall along AC, because $\angle\ D = \angle\ A$.
Let F and G be the points on AB and AC where C and E fall, and join FG.
Since $\angle\ AFG = \angle\ ABC$, $\therefore FG$ is $\parallel BC$; *I. 28*
$\therefore\ BF : FA = CG : GA$; *VI. 2*
$\therefore\ BA : FA = CA : GA$, by addition; *V. 18*
$\therefore\ BA : CA = FA : GA$, by alternation;
$= CD : ED$.
Similarly by applying $\triangle\ DCE$ to $\triangle\ ABC$ so that C falls on B, and again, so that E falls on C, it may be proved
that $AB : BC = DC : CE$,
and $AC : CB = DE : EC$.

PROPOSITION 5.

1. The eighth.
2. Let ABC be a triangle, H, K, L the middle points respectively of BC, CA, AB.
Then $AB =$ twice HK, $BC =$ twice KL, and $CA =$ twice LH;
$\therefore AB : BC = HK : KL$, $BC : CA = KL : LH$, and $CA : AB = LH : HK$;
$\therefore$ $\triangle$s ABC, HKL are similar.

3. Because $\triangle\ AGH$ is equiangular to $\triangle\ ABC$, *I.* 29
$\therefore\ BA : AC = GA : AH$. *VI.* 4
But $BA : AC = ED : DF$;
$\therefore\ GA : AH = ED : DF$.
Now $GA = ED$, $\therefore\ AH = DF$.
Similarly $GH = EF$;
$\therefore\ \triangle$s AGH, DEF are congruent, and mutually equiangular;
$\therefore\ \triangle$s ABC, DEF are mutually equiangular.

PROPOSITION 6.

1. The fourth.
2. Because $\triangle\ AGH$ is equiangular to $\triangle\ ABC$, *I.* 29
$\therefore\ BA : AC = GA : AH$.
But $BA : AC = ED : DF$;
$\therefore\ GA : AH = ED : DF$.
Now $GA = ED$; $\therefore\ AH = DF$;
$\therefore\ \triangle$s AGH, DEF are congruent, and mutually equiangular; *I.* 4
$\therefore\ \triangle$s ABC, DEF are mutually equiangular.
3. Because $BD : DA = AD : DC$, and $\angle$s BDA, ADC are right,
$\therefore\ \triangle$s BDA, ADC are similar; *VI.* 6
$\therefore\ \angle\ ABD = \angle\ CAD$, and $\angle\ BAD = \angle\ ACD$;
$\therefore\ \angle\ BAC = \angle\ ABD + \angle\ ACD$;
$\therefore\ \angle\ BAC$ is right.
Again because $CB : BA = AB : BD$,
and $\angle\ CBA = \angle\ ABD$,
$\therefore\ \triangle$s CBA, ABD are similar; *VI.* 6
$\therefore\ \angle\ BAC = \angle\ BDA$ = a right angle.
4. Join AF, AG.
Then $AD : DF = AE : EG$, and $\angle\ ADF = \angle\ AEG$; *I.* 29
$\therefore\ \triangle$s ADF, AEG are similar; *VI.* 6
$\therefore\ \angle\ DAF = \angle\ EAG$;
$\therefore\ AF$ falls along AG.
5. Because $AB : AC = AC : AD$,
$\therefore\ AB : AE = EA : AD$, and $\angle\ BAE = \angle\ EAD$;
$\therefore\ \triangle$s BAE, EAD are similar.
Hence $\angle\ ABE = \angle\ AED$.
But $\angle\ ACE = \angle\ AEC$; *I.* 5
$\therefore\ \angle\ ACE - \angle\ ABE = \angle\ AEC - \angle\ AED$;
$\therefore\ \angle\ BEC = \angle\ DEC$. *I.* 32

PROPOSITION 7.

1. Proposition A.
2. Because in the $\triangle$s ABC, DBA, $BC : CA = BA : AD$, and the angles ABC, DBA opposite to CA and AD are the same;
   $\therefore$ $\angle$s BAC, BDA opposite to BC and BA must either be equal or supplementary.
   Now $\angle$ BDA is right; $\therefore$ $\angle$ BAC is also right.

PROPOSITION 8.

1. Let B be the centre of a circle, and A any point on its $\bigcirc^{ce}$; let AD be drawn $\perp$ a radius, and let the tangent at A meet the radius produced at C. Join AB.
   Then $\angle$ BAC is right; *III.* 18
   $\therefore$ BA is a mean proportional between BD and BC. *VI.* 8, *Cor.*
2. Let A be the centre of a circle, EF a diameter. At E and F let there be drawn two tangents EB, FC, and let BC be another tangent to the circle at the point D. Join AB, AC, AD.
   Then $\triangle$s ABD, ABE are congruent; *I.* 8
   $\therefore$ $\angle$ BAD = half of $\angle$ EAD.
   Similarly $\angle$ CAD = half of $\angle$ FAD; $\therefore$ $\angle$ BAC is right.
   Now AD is $\perp$ BC; *III.* 18
   $\therefore$ AD is a mean proportional between BD and CD.
3. Because $\triangle$s BDA, BAC are similar,
   $\therefore BD : DA = BA : AC$.
   Because $\triangle$s ADC, BAC are similar,
   $\therefore AD : DC = BA : AC$;
   $\therefore \quad \left\{\begin{matrix} BD : DA \\ AD : DC \end{matrix}\right\} = \left\{\begin{matrix} BA : AC \\ BA : AC \end{matrix}\right\}$; *V.* 23, *Cor.*
   $\therefore BD : DC$ = duplicate of $BA : AC$. *V. Def.* 12, 13
4. Because $\angle$ $BAC = \angle$ BDA, and $\angle$ $ABC = \angle$ DBA,
   $\therefore$ $\triangle$s ABC, DBA are similar.
   Because $\angle$ $BAC = \angle$ AEC, and $\angle$ $ACB = \angle$ ECA,
   $\therefore$ $\triangle$s ABC, EAC are similar.
   Hence $\triangle$s DBA, EAC are similar.
   (1) follows from the similarity of $\triangle$s DBA, EAC;

(2) follows from the similarity of $\triangle$s ABC, DBA:
(3) and (4) follow from the similarity of $\triangle$s ABC, EAC.

When the vertical $\angle\ BAC$ is right, $\angle$s ADB, AEC are also right, and AD, AE coincide. The results (1), (2), (3), (4) of the deduction then become respectively (1), (2), (3), (4) of the Cor. to the proposition.

5. Let ABC be a triangle, and let AD, which is $\perp BC$, fall within the triangle (fig. to VI. 8).

(1) If $BD : DA = AD : DC$,
then $\triangle$s DBA, DAC are similar, *VI.* 6
and $\triangle\ ABC$ is right-angled. *I.* 32

(2) If $CB : BA = AB : BD$,
then $\triangle$s ABC, DBA are similar. *VI.* 6

(3) If $BC : CA = AC : CD$,
then $\triangle$s ABC, DAC are similar. *VI.* 6

(4) If $BC : BA = AC : AD$,
then $\triangle$s ABC, DAC are similar. *VI.* 7

Next let ABC be a triangle (fig. to the preceding deduction), and let AD, AE be drawn from A to the base BC, making $\angle\ ADB = \angle\ AEC$.

(1) If $BD : DA = AE : EC$,
then $\triangle$s DBA, EAC are similar, *VI.* 6
and each is similar to $\triangle\ ABC$. *I.* 32

(2) If $CB : BA = AB : BD$,
then $\triangle$s ABC, DBA are similar. *VI.* 6

(3) If $BC : CA = AC : CE$,
then $\triangle$s ABC, EAC are similar. *VI.* 6

(4) If $BC : BA = AC : AE$,
then $\triangle$s ABC, EAC are similar, if $\angle\ BAC = \angle\ AEC$,
and $\triangle$s ABC, EBA are similar, if $\angle\ BAC$ is supplementary to $\angle\ AEC$. *VI.* 7

PROPOSITION 9.

1. The tenth.
2. Make the construction given in the text, and let AE be three times AD.
3. Find the fifth of the given straight line, and measure off consecutively two other parts each equal to the fifth.

4. Cut off the required aliquot part from the base of the triangle, and join the end of the part to the vertex.
Cut off the required aliquot part from any side of the $\|^m$, and through the end of the part draw a parallel to the adjacent side of the $\|^m$. The proof follows from VI. 1.

5. Find four-sevenths of any side of the $\|^m$, and draw a parallel as in the preceding deduction.

PROPOSITION 10.

1. Join BC, and through D and E draw DF, EG each $\|$ BC, and meeting AB at F and G. Through D draw DHK $\|$ AB, and meeting GE and BC at H and K.
Because AGE is a triangle, $\therefore AF : FG = AD : DE$. *VI.* 2
Because DKC is a triangle, $\therefore DH : HK = DE : EC$. *VI.* 2
But $DH = FG$, and $HK = GB$; *I.* 34
$\therefore FG : GB = DE : EC$.

2. When K is less than L, the figure in the text will be modified in this manner. The points G and H, instead of being on the same side of the point A, will be on opposite sides; consequently BG, BH, instead of being on the same side of AB, will be on opposite sides; C will be nearer to A than to B, and D will be on BA produced. Though these modifications take place in the shape of the figure, no modification is required in the wording of either the construction or the proof.
When $K = L$, the point of external section D is infinitely distant on the line AB or BA.

3. Divide the base of the triangle internally in the given ratio, and join the point of section to the vertex.
Divide any side of the $\|^m$ internally in the given ratio, and through the point of section draw a parallel to the adjacent side of the $\|^m$.
The proof follows from VI. 1.

4. Let B and C be the two given points on the $\bigcirc^{ce}$.
Join BC, and divide it internally at D in the given ratio.
Bisect the arc BC at E; join ED and produce it to meet the $\bigcirc^{ce}$ at A. A is the point required.
Join AB, AC.
Since arc BE = arc EC, $\therefore \angle BAD = \angle DAC$; *III.* 27
$\therefore BA : AC = BD : DC$. *VI.* 3

PROPOSITION 11.

1. Yes. For if AB be less than AC, the third proportional to AB, AC will be greater than either of them, the third proportional to AC, AB will be less than either of them.
   Two third proportionals, one to AB, AC, and one to AC, AB.
2. Place AB, AC so as to contain any angle;
   produce AB, AC making $AD = AC$;
   join BC, and through D draw $DE \parallel BC$, meeting AC produced at E. AE is the third proportional.
   Because $\triangle$s ABC, ADE are mutually equiangular, *I.* 29
   $\therefore AB : AC = AD : AE$; *VI.* 4
   $\therefore AB : AC = AC : AE$.
3. (1) Let BD and DA (fig. to VI. 8) be the two given straight lines.
   Place BD and DA so as to contain a right angle BDA;
   join BA, and from A draw $AC \perp AB$ to meet BD produced at C. DC is the third proportional.
   For $BD : DA = AD : DC$. *VI.* 8, *Cor.*
   (2) Let AC and CD (fig. to VI. 13) be the two given straight lines.
   On AC as diameter describe a semicircle;
   with C as centre and CD as radius cut the arc of the semicircle at D. From D draw $DB \perp AC$.
   BC is the third proportional.
   Join AD, CD.
   Then $\triangle$ ADC is right-angled; *III.* 31
   $\therefore AC : CD = DC : CB$. *VI.* 8, *Cor.*
4. Join BC, DE.
   Then $\triangle$s ABC, ADE are mutually equiangular; *III.* 21
   $\therefore AB : AC = AD : AE$; *VI.* 4
   $\therefore AB : AC = AC : AE$.
5. Construct a rhombus $ABCD$, each of whose sides is equal to the second given straight line; produce BA to E, cutting off AE = the first given straight line.
   Join ED, and produce it to meet BC produced at F.
   CF is the third proportional.
   For $\triangle$s EAD, DCF are similar;
   $\therefore EA : AD = DC : CF$.

6. Let CB and BA be the two given straight lines.
Place them so as to contain any angle CBA, and join AC.
From A draw AD to BC, making $\angle\ ADB = \angle\ BAC$, by the second deduction from I. 31. BD is the third proportional.
For $CB : BA = AB : BD$, by (2) of the fourth deduction from VI. 8.

PROPOSITION 12.

1. VI. 11. When B and C are equal, the fourth proportional to A, B, C becomes the third proportional to A and B.

2. Yes. For suppose A to be greater than B.
When the three straight lines are taken in the order A, B, C, the fourth proportional will be less than C; when they are taken in the order B, A, C, the fourth proportional will be greater than C.
Three fourth proportionals. The order A, B, C, or A, C, B will give one fourth proportional; the order B, A, C, or B, C, A will give another; and the order C, A, B, or C, B, A will give the last.

3. Take two straight lines DG, DH containing any angle; from these cut off $DG = A$, $DH = B$, and $DE = C$;
join GH, and through E draw $EF \parallel GH$, meeting DH or DH produced at F. DF is the fourth proportional.
Because $\triangle$s DGH, DEF are mutually equiangular, *I.* 29
$\therefore\ DG : DH = DE : DF$; *VI.* 4
$\therefore\ A : B = C : DF$.

4. Let A, B, C be the three given straight lines.
From any point D draw DE, DF containing any angle, and respectively $= A$ and B.
Produce FD to G, making $DG = C$; describe a circle through the three points E, F, G, and produce ED to meet it at H.
DH is the fourth proportional.
Join EF, GH.
Then $\triangle$s DEF, DGH are mutually equiangular; *III.* 21
$\therefore\ DE : DF = DG : DH$; *VI.* 4
$\therefore\ A : B = C : DH$.

5. Let ABC be the given triangle, $K : L$ the given ratio.
To K, L and BC find a fourth proportional;

and from BC or BC produced, cut off BD = this fourth proportional. Join AD.

Then $K : L = BC : BD$,

$= \triangle ABC : \triangle ABD$. *VI.* 1

Similarly for the $\|^{m}$.

6. Let $K : L$ be the given ratio.

Through D draw $DE \parallel AB$, meeting AC at E.

To K, L and AE find a fourth proportional;

and from EC cut off EF = this fourth proportional.

Join FD, and produce it to meet AB at G.

Then $K : L = AE : EF$,

$= GD : DF$. *VI.* 2

PROPOSITION 13.

1. On AC as diameter, describe a semicircle;

from B draw $BD \perp AC$, and join CD.

CD is the mean proportional.

Join AD.

Then $AC : CD = DC : CB$. *VI.* 8, *Cor.*

2. Join AD, CE.

Then $\triangle$s ABD, EBC are mutually equiangular; *III.* 21

$\therefore AB : BD = EB : BC$. *VI.* 4

But BD and BE are equal; *III.* 3

$\therefore$ either of them is the mean proportional.

3. Join AD.

Then $\triangle$s ACD, DCB are mutually equiangular;

$\therefore AC : CD = DC : CB$. *VI.* 4

4. Let AB and BC (fig. to VI. 13) be the two straight lines, and let AB be greater than BC.

Construct BD the mean proportional as in the proposition, and join D to O, the centre of the semicircle ADC.

Then $\frac{1}{2}(AB + BC) = OD$;

and OD is greater than BD. *I.* 19, *Cor.*

5. Because $\triangle$s AED, BEF are mutually equiangular; *I.* 29

$\therefore DE : FE = AE : BE$. *VI.* 4

Now $DE : FE = \triangle AED : \triangle AEF$, *VI.* 1

and $AE : BE = \triangle AEF : \triangle BEF$; *VI.* 1

$\therefore \triangle AED : \triangle AEF = \triangle AEF : \triangle BEF$.

6. Let A and B be the two given straight lines.

Between A and B one mean C can be found. Between A and C one mean D can be found, and between C and B one mean E. Thus three means are found, the series being A, D, C, E, B.

By inserting a mean between A and D, D and C, C and E, E and B, 7 means are found between A and B; and by continuing the process, we may insert 15, 31, 63, &c. means.

The number of the means we can thus insert is always less by 1 than some power of 2. Hence the algebraical expression $2^n - 1$, where n may be any one of the natural numbers, will include all the numbers 1, 3, 7, 15, 31, 63, &c.

PROPOSITION 14.

1. (1) Join also FD, EG.

Because $\|^m AB = \|^m BC$, *Hyp.*

$\therefore \quad \triangle FBD = \triangle EBG$; *I.* 34

$\therefore \quad \triangle FED = \triangle EFG$;

$\therefore$ FE is $\|$ DG; *I.* 39

$\therefore$ $DB : BE = GB : BF$, by the fifth deduction from VI. 2.

(2) Because $DB : BE = GB : BF$,

$\therefore$ FE is $\|$ DG, by the fifth deduction from VI. 2.

$\therefore \triangle FED = \triangle EFG$; *I.* 37

$\therefore \triangle FBD = \triangle EBG$;

$\therefore \quad \|^m AB = \|^m BC$. *I.* 34

2. Let AF, CE meet at H; join HB.

Let HB, AD produced meet at K, and HB, CG produced meet at L.

Because EH, AK are parallel,

$\therefore$ $HB : BK = EB : BD$, by the fifth deduction from VI. 2,

$= FB : BG$, *VI.* 14

$= HB : BL$, by the fifth deduction from VI. 2,

since HF and CL are parallel.

Hence K and L are the same point.

3. Let AF, CE meet at H.

Then $DB : BE = GB : BF$; *VI.* 14

$\therefore \quad AF : FH = CE : EH$; *I.* 34

$\therefore \quad AC$ is $\|$ EF. *VI.* 2

Or thus, by joining AE, CF.

Because $\triangle\ AFE = \frac{1}{2} \parallel^m AB$, and $\triangle\ CEF = \frac{1}{2} \parallel^m BC$; *I.* 41

$\therefore\ \triangle\ AFE = \triangle\ CEF$;

$\therefore\ AC$ is $\parallel EF$. *I.* 39

4. Rectangles are mutually equiangular, and their bases and altitudes are the sides about the equal angles.

5. Let $ABCD$, $EFGH$ be equal $\parallel^{ms}$ having

$AB : EF = FG : BC$: to prove them mutually equiangular.

From A draw $AK \perp BC$, or BC produced,

and from E draw $EL \perp FG$, or FG produced.

Then $ABCD : EFGH = \left\{ \begin{matrix} BC : FG \\ AK : EL \end{matrix} \right\}$, by the eleventh deduction from VI. 1.

But $ABCD = EFGH$;

$\therefore\ AK : EL = FG : BC$, by (*),

$= AB : EF$;

$\therefore\ \triangle$s ABK, EFL are similar, and $\angle\ ABK = \angle\ EFL$. *VI.* 7

Now $\angle$s ABK, EFL are either both of them the angles contained by the reciprocal sides, or both of them the supplements of these angles, or one of them one of these angles, and the other the supplement of the other angle. In every case, therefore, the $\parallel^{ms}$ are mutually equiangular.

PROPOSITION 15.

1. Yes. Because $\triangle$s ABC, ADE are the halves of the $\parallel^{ms}$ whose adjacent sides are AB, AC, and AD, AE.

2. (1) Because $\triangle\ BCE = \triangle\ DCE$,

$\therefore\ BD$ is $\parallel CE$; *I.* 39

$\therefore\ AC : AD = AE : AB$, by the fifth deduction from VI. 2.

(2) Because $AC : AD = AE : AB$,

$\therefore\ BD$ is $\parallel CE$, by the fifth deduction from VI. 2;

$\therefore\ \triangle\ BCE = \triangle\ DCE$; *I.* 37

$\therefore\ \triangle\ ABC = \triangle\ ADE$.

3. Equal triangles which have one angle of the one supplementary to one angle of the other, have their sides about the supplementary angles reciprocally proportional.

* If two ratios, when compounded together, give a ratio of equality, the one ratio is the reciprocal of the other.

Conversely: Triangles which have one angle of the one supplementary to one angle of the other, and their sides about the supplementary angles reciprocally proportional, are equal.

4. Because $BD : DC = DC : DE$,
and $AD : DB = BD : DC$; *VI.* 8, *Cor.*
$\therefore$ $AD : DB = DC : DE$;
and $\angle ADE = \angle BDC$;
$\therefore$ $\triangle ADE = \triangle BDC$. *VI.* 15

5. The proof of this is similar to the proof of the fifth deduction from VI. 14, and may be obtained from it by leaving out the letters D and H, substituting triangles for $\|^{ms}$, and omitting the last sentence.

6. Because BE is $\parallel AC$. *I.* 28
$\therefore$ $BD : DC = ED : DA$, by the fifth deduction from *VI.* 2;
and $\angle BDA = \angle CDE$;
$\therefore$ $\triangle ABD = \triangle ECD$. *VI.* 15

7. Let ABC be the given triangle.
Divide AB internally at D in medial section, so that AD is the greater segment. Through D draw $DE \parallel BC$, and join DC.
Because $AB \cdot BD = AD^2 = AD \cdot AD$,
$\therefore$ $AB : AD = AD : BD$, by the fourth deduction from VI. 14.
But $AB : AD = BC : DE$; *VI.* 4
$\therefore$ $AD : BD = BC : DE$;
and $\angle ADE = \angle DBC$;
$\therefore$ $\triangle ADE = \triangle DBC$. *VI.* 15
If AB be divided externally in medial section at D', so that D' is on BA produced, and through D' there be drawn $D'E'$ to meet CA produced at E', and $D'C$ be joined, then $\triangle AD'E' = \triangle D'BC$.
The proof is the same as the preceding, with D' and E substituted for D and E.

PROPOSITION 16.

1. The results (1), (2), (3), (4) follow immediately from (1), (2), (3), (4) of VI. 8, Cor., by application of VI. 16.

2. $AB^2 + AC^2 = CB \cdot BD + BC \cdot CD$,
$= BC^2$. *II.* 2

3. In Euclid's proof of I. 47, it is proved that $\|^m$ BL = square BG, and $\|^m$ CL = square CH;
   and these are results (2) and (3) of the first deduction.
4. By the first and second deductions from III. 21,
   $\triangle$s AEB, DEC (figs. to III. 35 and Cor.) are mutually equiangular;
   $\therefore\ AE : EB = DE : EC;$ *VI.* 4
   $\therefore\ AE : EC = BE \cdot ED.$ *VI.* 16
5. The results (1), (2), (3), (4) follow immediately from (1), (2), (3), (4) of the fourth deduction from VI. 8, by application of VI. 16.
6. $AB^2 + AC^2 = CB \cdot BD + BC \cdot CE,$
   $= BC \cdot (BD + CE),$
   $= BC \cdot (BC \pm DE),$
   $= BC^2 \pm BC \cdot DE.$
7. When $\angle\ BAC$ is right, AD and AE coincide;
   $\therefore\ DE$ vanishes, and so does $BC \cdot DE$.
8. This is the fourth deduction over again.
9. Let $\triangle\ ABC$ be right-angled at A, and in it let there be inscribed the square $DEFG$, D and G being on AB, AC respectively: to prove $DEFG = BE \cdot FC$.
   Because $\angle\ BDE$ = complement of $\angle\ B$,
   $= \angle\ C;$
   $\therefore$ $\triangle$s BED, GFC are mutually equiangular;
   $\therefore\ BE : ED = GF : FC;$ *VI.* 4
   $\therefore\ BE \cdot FC = ED \cdot GF$ = square $DEFG$. *VI.* 16

PROPOSITION 17.

1. Of the last proposition.
2. Let AB (fig. to VI. 30) be divided so that $AB : AC = AC : BC;$
   then $AB \cdot BC = AC^2.$ *VI.* 17
   The converse is VI. 30.
3. From the similar $\triangle$s AEG, CEB there results
   $AE : CE = GE : BE;$ *VI.* 4
   from the similar $\triangle$s AEB, CEF there results
   $AE : CE = BE : FE;$ *VI.* 4
   $\therefore\ GE : BE = BE : FE;$
   $\therefore\ GE \cdot FE = BE^2.$ *VI.* 17

PROPOSITION 18.

1. In general, as many polygons as $CDEF$ has sides. For the constructed polygon $ABHG$ might have AB homologous to CD, to DE, to EF, or to FC.

2. (*a*) The polygons $ABHG$ and $CDEF$ are mutually equiangular, because their corresponding sides are parallel and drawn in the same directions; and they can be proved to have their sides proportional exactly as in the text.

(*b*) From the similar $\triangle$s OAG, OCF,
$AG : CF = OA : OC$; *VI. 4*
from the similar $\triangle$s OAB, OCD,
$AB : CD = OA : OC$; *VI. 4*
$\therefore$ $AG : CF = AB : CD$;
$\therefore$ $AG : AB = CF : CD$, by alternation.
Similarly $AB : BH = CD : DE$;
$\therefore$ $AG : BH = CF : DE$, by direct equality;
and $AG : CF = BH : DE$, by alternation.
Now $AG : CF = OG : OF$,
and $BH : DE = OH : OE$;
$\therefore$ $OG : OF = OH : OE$;
$\therefore$ FE is $\parallel$ GH, by the fifth deduction from VI. 2.
Hence in like manner $BH : HG = DE : EF$,
and $HG : GA = EF : FC$.
And the polygons are mutually equiangular; *I. 34, Cor.*
$\therefore$ they are similar.

3. Because $BL : BA = BM : BG$,
$\therefore$ LM is $\parallel$ AG, by the fifth deduction from VI. 2.
Similarly MN is $\parallel$ GH;
$\therefore$ $BLMN$ is equiangular to $BAGH$. *I. 34, Cor.*
Because $\triangle$s BLM, BAG are similar,
$\therefore$ $BL : LM = BA : AG$,
and $LM : MB = AG : GB$. *VI. 4*
Because $\triangle$s BMN, BGH are similar,
$\therefore$ $MB : MN = GB : GH$;
$\therefore$ $LM : MN = AG : GH$, by direct equality;
and $MN : NB = GH : HB$;
$\therefore$ $BLMN$ is similar and similarly situated to $BAGH$.

4. Produce AB, GB, HB through B, and on AB produced take any point L. Find M on GB produced such that $BA : BL = BG : BM$, and N on HB produced such that $BG : BM = BH : BN$; *VI.* 12
and join LM, MN.

The proof that $BLMN$ and $BAGH$ are similar is the same as before.

PROPOSITION 19.

1. Let $BC : EF = EF : BG$ (fig. to VI. 19), and on BC, EF let there be two $\triangle$s ABC, DEF similar and similarly described.

Because $BC : EF = EF : BG$,

$\therefore\ BC : BG =$ duplicate of $BC : EF$, *V. Def.* 13, *Cor.*
$= \triangle ABC : \triangle DEF$. *VI.* 19

2. Because $DE : EF = AB : BC$,

$\therefore\ DE : AB = EF : BC$, by alternation,
$= BC : EG$; *Const.*

$\therefore\ \triangle ABC = \triangle DEG$. *VI.* 15

Because $EF : BC = BC : EG$,

$\therefore\ EF : EG =$ duplicate of $EF : BC$; *V. Def.* 13, *Cor.*

$\therefore\ \triangle DEF : \triangle DEG =$ duplicate of $EF : BC$;

$\therefore\ \triangle DEF : \triangle ABC =$ duplicate of $EF : BC$.

3. Triangles ABC, HBG are similar, and $\triangle$s HBG, DEF are congruent;

$\therefore\ \triangle ABC : \triangle ABG = BC : BG$, *VI.* 1
and $\triangle ABG : \triangle HBG = AB : HB$, *VI.* 1
$= BC : BG$; *VI.* 4

$$\therefore \begin{Bmatrix} \triangle ABC : \triangle ABG \\ \triangle ABG : \triangle HBG \end{Bmatrix} = \begin{Bmatrix} BC : BG \\ BC : BG \end{Bmatrix};$$

$\therefore\ \triangle ABC : \triangle HBG =$ duplicate of $BC : BG$;

$\therefore\ \triangle ABC : \triangle DEF =$ duplicate of $BC : EF$.

4. (1) Let ABC, DEF be two similar triangles, and let AH, DK be a pair of corresponding medians.

Because $AB : BC = DE : EF$,

$\therefore\ AB : BH = DE : EK$;

and $\angle ABH = \angle DEK$;

$\therefore$ $\triangle$s ABH, DEK are similar, *VI.* 6

and $AB : DE = AH : DK$.

But $\triangle\ ABC : \triangle\ DEF =$ duplicate of $AB : DE$;
$\therefore\ \triangle\ ABC : \triangle\ DEF =$ duplicate of $AH : DK$.

(2) Let ABC, DEF be two similar triangles, and let AL, DM be a pair of corresponding altitudes.

Then $\triangle$s ABL, DEM are similar, *VI.* 4
and $AB : DE = AL : DM$.
But $\triangle\ ABC : \triangle\ DEF =$ duplicate of $AB : DE$;
$\therefore\ \triangle\ ABC : \triangle\ DEF =$ duplicate of $AL : DM$.

(3) Let ABC, DEF be two similar triangles, and I and J be their inscribed centres. Join IB, IC, JE, JF;
draw $IP \perp BC$, and $JQ \perp EF$.

Then BI, CI bisect $\angle$s ABC, ACB,
and EJ, FJ bisect $\angle$s DEF, DFE;
$\therefore\ \triangle$s IBC and JEF are similar, *VI.* 4
and $BC : EF = IP : JQ$, by (2).
But $\triangle\ ABC : \triangle\ DEF =$ duplicate of $BC : EF$;
$\therefore\ \triangle\ ABC : \triangle\ DEF =$ duplicate of $IP : JQ$.

(4) Let ABC, DEF be two similar triangles, and S and T be their circumscribed centres. Draw SL, SK respectively $\perp AB$, AC, and TN, TM respectively $\perp DE$, DF.

Then SL, SK bisect AB, AC, and TN, TM bisect DE, DF;
$\therefore\ AL : AK = DN : DM$;
and $\angle\ LAK = \angle\ NDM$;
$\therefore\ \triangle$s ALK and DNM are similar. *VI.* 6
Because $\angle\ ALS = \angle\ DNT$, and $\angle\ ALK = \angle\ DNM$,
$\therefore\ \angle\ SLK = \angle\ TNM$.
Similarly $\angle\ SKL = \angle\ TMN$;
$\therefore\ \triangle$s SLK and TNM are similar.
Hence it may be proved, as in VI. 20, that the quadrilaterals $AKSL$, $DMTN$ have their sides proportional;
and they are mutually equiangular;
$\therefore$ they are similar;
$\therefore\ \triangle$s ALS, DNT may be proved to be similar;
$\therefore\ 2\,AL : 2\,DN = AS : DT$;
$\therefore\ AB : DE = AS : DT$.
Hence $\triangle\ ABC : \triangle\ DEF =$ duplicate of $AS : DT$.

PROPOSITION 20.

1. For squares are similar polygons, since they are mutually equiangular, and their sides are to each other in a ratio of equality.
2. For polygon $ABCDE$: polygon $FGHKL$ (fig. to VI. 20),
$= $ duplicate of $AB : FG$,
$= AB^2 : FG^2$, by the last deduction,
$= BE^2 : GL^2, = $ &c.
3. For $AB : FG = BC : GH = CD : HK = DE : KL = EA : LF$;
$\therefore\ AB : FG = AB + BC + CD + DE + EA : FG + GH + HK + KL$.
4. This has been shown in (a) of the second deduction from VI. 18.
5. Let $ABCD$, $A'B'C'D'$ be two similar polygons. Take any point O inside $ABCD$, and join OA, OB, OC. At A' make $\angle\ B'A'O' = \angle\ BAO$, and at B' make $\angle\ A'B'O' = \angle\ ABO$; let $A'O'$, $B'O'$ meet at O', and join $O'C'$.
Then $\triangle$s OAB, $O'A'B'$ are similar; *VI.* 4
$\therefore\ OB : AB = O'B' : A'B'$.
But $AB : BC = A'B' : B'C'$;
$\therefore\ OB : BC = O'B' : B'C'$, by equality.
And $\angle\ ABC - \angle\ ABO = \angle\ A'B'C' - \angle\ A'B'O'$;
$\therefore\ \angle\ OBC = \angle\ O'B'C'$;
$\therefore$ $\triangle$s OBC, $O'B'C'$ are similar.
In like manner $\triangle$s OCD, $O'C'D'$ are similar, as well as $\triangle$s ODA, $O'D'A'$.
6. The point O could be taken outside the polygon $ABCD$, or on one of its sides, and the point O', homologous to O with respect to the other polygon, would be found to be outside, or on the corresponding side.
7. It will be sufficient to take the case of two similar and similarly situated triangles.
Let ABC, $A'B'C'$ be two similar and similarly situated triangles; let BB' produced meet AA' produced at O, and if CC' produced does not meet AA' produced at O, let it meet it at O'.
Then from the similar $\triangle$s OAB, $OA'B'$,
we have $OA : OA' = AB : A'B'$; *VI.* 4
from the similar $\triangle$s ABC, $A'B'C'$

we have $AB : A'B' = AC : A'C'$; VI. 4

$\therefore \quad OA : OA' = AC : A'C'$.

But from the similar $\triangle$s $O'AC$, $O'A'C'$

we have $O'A : O'A' = AC : A'C'$; VI. 4

$\therefore \quad OA : OA' = O'A : O'A'$;

$\therefore$ O and O' are the same point.

[If the last step be considered to require more proof, consult the first deduction of Book VI.]

The same proof is applicable when $\triangle$s ABC, $A'B'C'$ are similar and oppositely situated.

8. Let ABC be the given triangle.

On BC describe, outwardly from the triangle, the square $BDEC$. Join AD, AE cutting BC at D', E'.

Through D' draw $D'B' \parallel DB$ and meeting AB in B'; through E' draw $E'C' \parallel EC$ and meeting AC in C'.

Join $B'C'$. $B'D'E'C'$ is the square required.

Three squares may be inscribed in the triangle ABC. The two others are obtained from the squares described on CA and AB.

9. Let ABC be the given triangle, $DEFG$ the given rectangle.

On BC describe, outwardly from the triangle, the rectangle $BHKC$ similar to the rectangle $DEFG$, and such that BC and DG are homologous sides. Join AH, AK cutting BC at H', K'. Through H' draw $H'B' \parallel HB$ and meeting AB in B'; through K' draw $K'C' \parallel KC$ and meeting AC in C'.

Join $B'C'$. $B'H'K'C'$ is the rectangle required.

Another rectangle $BHKC$ can be described on BC similar to $DEFG$, if BC be made homologous to DE; whence another rectangle $B'H'K'C'$ is obtained.

Hence six rectangles can be inscribed in the triangle ABC similar to $DEFG$. The two other pairs are obtained from the rectangles described on CA and AB.

PROPOSITION 21.

If BC be not $= FG$, let BC be greater than FG.

Take KL a third proportional to BC, FG.

Then $BC : FG = FG : KL$.

Now since BC is greater than FG,

$\therefore$ FG is greater than KL;

$\therefore$ BC is greater than KL.

But $BC : KL = ABCD : EFGH$; *VI. 20, Cor.*

$\therefore$ $ABCD$ is greater than $EFGH$, which is impossible.

PROPOSITION 22.

1. For AB^2 and CD^2 are similar polygons described on AB and CD; and EF^2 and GH^2 are similar polygons described on EF and GH.
2. Let $AB : CD = EF : GH$:

   to prove duplicate of $AB : CD$ = duplicate of $EF : GH$.

   Take X a third proportional to AB, CD,

   and O a third proportional to EF, GH;

   and prove as in the proposition that $AB : X = EF : O$.

   Now $AB : X$ = duplicate of $AB : CD$,

   and $EF : O$ = duplicate of $EF : GH$.

PROPOSITION 23.

1. In the figure to the proposition, join DF, FE, EG.

   The construction and proof of the theorem, that triangles which have one angle of the one equal to one angle of the other are to one another in the ratio compounded of the ratios of the sides about those angles, are obtained from the construction and proof of the proposition in the text by substituting for $\parallel^{ms}$ AB, FE, BC respectively, $\triangle$s DBF, FBE, EBG.

   To prove the other part of the deduction, join AB.

   Then $\triangle\, BDA = \triangle\, DBF$, and $AD = FB$.

   But $\triangle\, DBF : \triangle\, EBG = \left\{\begin{matrix} DB : BE \\ FB : BG \end{matrix}\right\}$;

   $\therefore$ $\triangle\, BDA : \triangle\, EBG = \left\{\begin{matrix} DB : BE \\ AD : BG \end{matrix}\right\}$.

2. Because $\parallel^{m}\, AB : \parallel^{m}\, BC = \left\{\begin{matrix} DB : BE \\ FB : BG \end{matrix}\right\}$,

   and $\parallel^{m}\, AB = \parallel^{m}\, BC$;

   $\therefore$ $DB : BE = GB : BF$.

Because $\|^m AB : \|^m BC = \left\{\begin{matrix} DB : BE \\ FB : BG \end{matrix}\right\}$,

and $DB : BE = GB : BF$;

$\therefore \left\{\begin{matrix} DB : BE \\ FB : BG \end{matrix}\right\}$ is a ratio of equality;

$\therefore \|^m AB = \|^m BC$.

3. Let the $\|^{ms}$ AB, BC be rectangles.

Then $\|^m AB : \|^m BC = \left\{\begin{matrix} DB : BE \\ FB : BG \end{matrix}\right\}$;

$\therefore DB \cdot BF : EB \cdot BG = \left\{\begin{matrix} DB : BE \\ FB : BG \end{matrix}\right\}$.

4. Mutually equiangular $\|^{ms}$ are to one another as the rectangles contained by pairs of adjacent sides.

Triangles which have one angle of the one equal or supplementary to one angle of the other are to one another as the rectangles contained by the sides about those angles.

5. From the similar $\triangle$s ABK, EBH,

$AK : EH = AB : EB$.

But $\|^m AC : \|^m EF = \left\{\begin{matrix} AK : EH \\ BC : BF \end{matrix}\right\}$, by the eleventh deduction from VI. 1;

$\therefore \quad \|^m AC : \|^m EF = \left\{\begin{matrix} AB : EB \\ BC : BF \end{matrix}\right\}$.

6. Because $\triangle ABC : \triangle DEF = \left\{\begin{matrix} AB : DE \\ BC : EF \end{matrix}\right\}$,

and $AB : DE = BC : EF$;

$\therefore \quad \left\{\begin{matrix} AB : DE \\ BC : EF \end{matrix}\right\}$ = duplicate of $BC : EF$;

$\therefore \triangle ABC : \triangle DEF$ = duplicate of $BC : EF$.

PROPOSITION 24.

1. Because $\|^{ms}$ $AEFG$, $FHCK$ are similar,

$\therefore \quad EF : FG = HC : CK$, VI. 24

$\qquad = FK : FH$; I. 34

and $\angle EFH = \angle KFG$;

$\therefore \quad \|^m EH = \|^m GK$. VI. 14

2. For $\|^m EG : \|^m BF = GF : FH$, VI. 1

$\qquad = \|^m FD : \|^m HK$. VI. 1

3. Because $\|^{ms}$ $AEFG$, $ABCD$ are similar,

$\therefore$ $AE : AB = AG : AD$;

$\therefore$ EG is $\|$ BD, by the fifth deduction from VI. 2.

Similarly HK is $\|$ BD.

PROPOSITION 25.

1. Let D be the given square.

Make any equilateral $\triangle$ ABC, on BC describe a rectangle $BLEC = \triangle ABC$, and on CE describe a rectangle $CEMF = D$. Between BC and CF find a mean proportional GH; *VI.* 13 on GH describe an equilateral $\triangle$ KGH.

Then $BC : CF = BLEC : CEMF$, *VI.* 1

$= ABC : D$.

But because $BC : GH = GH : CF$,

and because ABC, KGH are similar and similarly described;

$\therefore$ $BC : CF = ABC : KGH$. *VI.* 20, *Cor.*

Hence $ABC : D = ABC : KGH$;

$\therefore$ $KGH = D$.

2, 3, 4, 5, 6. It has not been deemed necessary to write out the solutions of these deductions, as they are similar to the preceding one, which itself is little more than a transcript of the proposition.

7. Let $ABCD$ be the given polygon, and let the straight line F be the given perimeter.

Find a straight line E = the perimeter of $ABCD$;

to E, F, and AB, a side of the given polygon, find a fourth proportional GH; *VI.* 12

On GH describe a polygon $GHKL$ similar to $ABCD$. *VI.* 18

Then perimeter of $ABCD$: perimeter of $GHKL = AB : GH$, by the third deduction from VI. 20;

$\therefore$ E : perimeter of $GHKL = AB : GH$.

But $E : F = AB : GH$.

$\therefore$ perimeter of $GHKL = F$.

8. Let $ABCD$ be the given polygon, $K : L$ the given ratio.

On AB describe a square, and from AB or AB produced, cut off AM, a fourth proportional to K, L, and AB. *VI.* 12

Then $AB^2 : AB \cdot AM = AB : AM$, *VI.* 1

$= K : L$.

Find PQ the side of a square, such that $PQ^2 = AB \cdot AM$, *II.* 14

and on PQ describe a polygon $PQRS$ similar to $ABCD$. *VI.* 18

$$\begin{aligned} \text{Because } AB : PQ &= AB : PQ, \\ \therefore\ ABCD : PQRS &= AB^2 : PQ^2, \qquad VI.\ 22 \\ &= AB^2 : AB \cdot AM, \\ &= K : L. \end{aligned}$$

9. Let P be the point inside the circle, $K : L$ the given ratio. Construct a first rectangle, whose sides shall be in the ratio of $K : L$, and a second rectangle equal to the potency of P with respect to the circle (see p. 209 of *Euclid*). Describe a third rectangle similar to the first, and equal to the second; and, by the fourth deduction from III. 18, draw through P a chord, whose length is the sum of two adjacent sides of the third rectangle.

PROPOSITION 26.

1. $\|^{ms}$ $ABCD$, $AEHK$ are similar and similarly situated; *VI.* 24
   $\therefore\ BA : AD = EA : AK.$
   But $BA : AD = EA : AG$;
   $\therefore\ EA : AK = EA : AG$, which is impossible.
   Hence AC must pass through F.
2. Produce BA, DA, through A.
   On BA produced take any point E, and find AG such that $BA : AD = EA : AG$. *VI.* 12
   Through E and G draw $EF \parallel BC$ and $GF \parallel DC$.
   Then $\|^{ms}$ $ABCD$, $AEFG$ may be proved to be similar; and it may be shown, as in the proposition, that $\angle BAC = \angle EAF$;
   $\therefore$ the diagonals AC and AF are in the same straight line by the sixth deduction from I. 15.
3. Let $ABCD$ (fig. to VI. 26) be the given $\|^m$.
   Construct, by the eighth deduction from VI. 25, a second $\|^m$ similar to $ABCD$, and having to it the given ratio.
   From AB cut off AE, equal to that side of the second $\|^m$ which is homologous to AB; and from AD cut off AG, equal to that side of the second $\|^m$ which is homologous to AD.
   Through E draw $EF \parallel AD$, and through G draw $GF \parallel AB$.
   $AEFG$ is the required $\|^m$.

PROPOSITION 27.

1. If the figure be made and lettered to correspond with that in the text, the proof will be:

Because $BD = DC$, $\therefore$ $FE = EL$; *I.* 34

$\therefore$ $\|^m\ KE = \|^m\ EN$. *I.* 36

Again $\|^m\ MN = \|^m\ DH$, by the second deduction from I. 43.

But $\|^m\ EN$ is less than $\|^m\ MN$;

$\therefore$ $\|^m\ KE$ is less than $\|^m\ DH$.

From KD take each of these unequals;

then $\|^m\ BE$ is greater than $\|^m\ BH$.

2. The proposition then becomes the second part of the first deduction from II. 5, or the ninth deduction from II. 5.

PROPOSITION 28.

1. Write down word for word, and letter for letter, the proof of the proposition, but substitute E' for E.

2. Let F be the middle point of AB.

Because $AE \cdot EB = FB^2 - FE^2$, *II.* 5

and $AE' \cdot E'B = FA^2 - FE'^2$, *II.* 5

$= FB^2 - FE'^2$;

$\therefore$ $FB^2 - FE^2 = FB^2 - FE'^2$;

$\therefore$ $FE^2 = FE'^2$, and $FE = FE'$;

$\therefore$ $AE' = BE$, and $BE' = AE$.

3. The rectangle $K \cdot L$ may be as small as we please, but it must not be greater than the square on half AB.

4. Let AB be the given straight line.

Convert the rectangle $K \cdot L$ into a square, *II.* 14

and let the straight line C be a side of it.

Bisect AB at D; draw $DE \perp AB$, and $= C$.

With E as centre, and radius $= AD$ or DB, cut AB at F; and join EF.

Then $AF \cdot FB = DB^2 - DF^2$, *II.* 5

$= EF^2 - DF^2$,

$= DE^2$, *I.* 47, *Cor.*

$= C^2$.

PROPOSITION 29.

1. Write down word for word, and letter for letter, the proof of the proposition, but substitute E' for E.

2. Let F be the middle point of AB.

Because $AE \cdot EB = FE^2 - FB^2$, *II. 6*

and $AE' \cdot E'B = FE'^2 - FA^2$, *II. 6*

$= FE'^2 - FB^2$;

$\therefore FE^2 - FB^2 = FE'^2 - FB^2$;

$\therefore FE^2 = FE'^2$, and $FE = FE'$;

$\therefore AE' = BE$, and $BE' = AE$.

3. The rectangle $K \cdot L$ may be as small or as large as we please.

4. Let AB be the given straight line.

Convert the rectangle $K \cdot L$ into a square, *II. 14*

and let the straight line C be a side of it.

Bisect AB at D; draw $BE \perp AB$, and $= C$.

Join DE, and with D as centre, and DE as radius, cut AB produced at F.

Then $AF \cdot FB = DF^2 - DB^2$, *II. 6*

$= DE^2 - DB^2$,

$= BE^2$, *I. 47, Cor.*

$= C^2$.

PROPOSITION 30.

1. Because $BC : BD = BD : DC$.

$\therefore BD^2 = BC \cdot DC$ *VI. 17*

$= AC^2$, by (3) of the first deduction from VI. 16;

$\therefore BD = AC$.

Conversely: Since $BD = AC$, $\therefore BD^2 = AC^2$.

But $AC^2 = BC \cdot CD$, by (3) of the first deduction from VI. 16

$\therefore BD^2 = BC \cdot CD$;

$\therefore BC : BD = BD : CD$. *VI. 17*

2. Because $AC : CB = DF : FE$,

$\therefore AB : CB = DE : FE$, by addition.

But $AB : CB = AB \cdot BC : CB^2$, *VI.* 1
and $DE : FE = DE \cdot EF : FE^2$; *VI.* 1
$\therefore \quad AB \cdot BC : CB^2 = DE \cdot EF : FE^2$;
$\therefore \; AB \cdot BC : DE \cdot EF = CB^2 : FE^2$,
$= AC^2 : DF^2$.
Hence if $AB \cdot BC = AC^2$, $DE \cdot EF = DF^2$.

PROPOSITION 31.

1. Since squares are similar rectilineal figures, let X, Y, Z be squares.
   Now $Y + Z = X$; *VI.* 31
   $\therefore \; AB^2 + AC^2 = BC^2$.
2. No.
3. Because $AB : BC = AB : BC$,
   $\therefore \quad AB^2 : BC^2 = Y : X$. *VI.* 22
   Because $AC : BC = AC : BC$,
   $\therefore \quad AC^2 : BC^2 = Z : X$; *VI.* 22
   $\therefore \quad AB^2 + AC^2 : BC^2 = Y + Z : X$. *V.* 24
   But $AB^2 + AC^2 = BC^2$; *I.* 47
   $\therefore \quad Y + Z = X$.
4. The arc of the semicircle described on BC will pass through A; and since the semicircle on BC = the semicircle on AB + the semicircle on AC, if the segments of the semicircle on BC, which AB and AC cut off from it, be taken away, there remain the two lunules = $\triangle\ ABC$.

PROPOSITION 32.

1. For $BA : AC = CD : DF$; AB is $\parallel$ DC, and AC is $\parallel$ DF, but BC and CF are not in the same straight line.
2. Join CE, DF.
   Because $AC : BD = AE : BF$,
   $\therefore \quad AC : AE = BD : BF$, by alternation;
   and $\angle\ CAE = \angle\ DBF$; *I.* 34, *Cor.*
   $\therefore$ $\triangle$s CAE, DBF are similar; *VI.* 6
   $\therefore$ CE is $\parallel$ DF;
   $\therefore$ BA, DC, EF are concurrent, by the seventh deduction from VI. 20.

3. Because AC is $\parallel$ BD, and CH is $\parallel$ DL,
$\therefore$ $\triangle$s ACH, BDL are similar; *VI.* 4
$\therefore \quad AH : BL = AC : BD,$
$= AE : BF.$
Now $\angle EAH = \angle FBL$;
$\therefore$ $\triangle$s AHE, BLF are similar, *VI.* 6
and $\angle AHE = \angle BLF$;
$\therefore$ EH is $\parallel$ FL.

PROPOSITION 33.

On the same chord AB let there be two arcs ACB, ADB (fig. to III. 23), and let ADC, AFE be two straight lines drawn from A, and cutting the arcs at C, D, and E, F respectively.
Then arc BE : arc $EC = \angle BAE : \angle EAC$, *VI.* 33
$= \angle BAF : \angle FAD$,
$=$ arc BF : arc FD. *VI.* 33

PROPOSITION B.

1. Because $\angle BAE = \angle CAE$, $\therefore$ arc BE = arc CE. *III.* 26
Now if AE be a diameter, arc AB = arc AC;
and $\triangle$ ABC is isosceles.
2. This is the first deduction of Book II.
3. No. Because the bisector of the interior vertical angle, which is $\perp$ the bisector of the exterior vertical angle, would then become a tangent to the circle.
4. Because $\angle DCE = \angle BAE$, *III.* 21
$= \angle CAE$,
and $\angle AEC = \angle CED$;
$\therefore$ $\triangle$s AEC, CED are similar;
$\therefore AE : EC = CE : ED$, *VI.* 4
$\therefore AE \cdot ED = CE^2$.
5. Let ABC be a triangle having the vertical $\angle BAC$ bisected by AD, and the exterior vertical angle bisected by AD';
then $BD : DC = BA : AC$, *VI.* 3
and $BD' : D'C = BA : AC$; *VI.* A
$\therefore$ BC is cut internally and externally in the same ratio.

Because AD' is $\perp AD$,

$\therefore\ DD'^2 = AD^2 + AD'^2,$ *I.* 47

$= AB \cdot AC - BD \cdot DC + BD' \cdot D'C - AB \cdot AC,$ *VI.* B

$= BD' \cdot D'C - BD \cdot DC.$

6. If ABC be a triangle, and AD be drawn to the base so that $AD^2 = AB \cdot AC - BD \cdot DC$, then AD will bisect $\angle\ BAC$. If AD do not bisect $\angle\ BAC$, then $AB = AC$.

Make the same construction as in the proposition.

Because $AD^2 = AB \cdot AC - BD \cdot DC,$

$\therefore\ AD^2 + BD \cdot DC = AB \cdot AC;$

$\therefore\ AD^2 + AD \cdot DE = AB \cdot AC;$ *III.* 35

$\therefore\ AD \cdot AE = AB \cdot AC;$ *II.* 3

$\therefore\ AB : AD = AE : AC.$ *VI.* 16

Now in $\triangle$s ABD, AEC, since $\angle$s ABD, AEC, which are opposite AD, AC, one pair of homologous sides, are equal; *III.* 21

$\therefore$ $\angle$s BDA, ECA opposite AB, AE, the other pair of homologous sides, are either equal or supplementary. *VI.* 7

If $\angle\ BDA = \angle\ ECA$, then $\angle\ BAD = \angle\ EAC$, *I.* 32, *Cor.* 1 and AD bisects $\angle\ BAC$.

If $\angle\ BDA$ be supplementary to $\angle\ ECA$, since $\angle\ BDA$ = an angle at the $\bigcirc^{ce}$ standing on the sum of the arcs AB, CE, by the eighth deduction from III. 26, and III. 20, and $\angle\ ECA$ stands on the arc ABE;

$\therefore$ arc AB + arc CE + arc ABE = the whole $\bigcirc^{ce}$, by the sixth deduction from III. 26;

$\therefore$ arc AB = arc AC, and $AB = AC$.

The proof of the converse of the second part of the proposition is similar to the preceding. In this case it may be shown that $\angle$s BDA, ECA are always acute, and therefore cannot be supplementary. It may also be added that when $\triangle\ ABC$ is isosceles, and AD is drawn to the base produced, it is impossible to have $AD^2 = BD \cdot DC - AB \cdot AC$, for then $AD^2 = BD \cdot DC + AB \cdot AC$.

See the first deduction of Book II.

7. $AD^2 = AB \cdot AC - BD \cdot DC,$

$= cb - \dfrac{ac}{c+b} \cdot \dfrac{ab}{c+b}$, by the ninth deduction from VI. 3,

$$= \frac{bc\{(b+c)^2 - a^2\}}{(c+b)^2} = \frac{bc(b+c+a)(b+c-a)}{(c+b)^2},$$

$$= \frac{4\,bc \cdot s(s-a)}{(c+b)^2}.$$

$$AD^2 = BD \cdot DC - AB \cdot AC,$$

$$= \frac{ac}{c-b} \cdot \frac{ab}{c-b} - cb, \text{ by the third deduction from VI. A,}$$

$$= \frac{bc\{a^2 - (c-b)^2\}}{(c-b)^2} = \frac{bc(a+c-b)(a-c+b)}{(c-b)^2},$$

$$= \frac{4\,bc\,(s-b)(s-c)}{(c-b)^2}.$$

8. (1) Let ABC (fig. to VI. 3) be the triangle required. To obtain it we know AB, AC, AD.

Through C draw $CE \parallel DA$, meeting BA produced at E.

Then, as in VI. 3, $AE = AC$; $\therefore$ AE is known;

$\therefore$ BE is known.

Now from the similar $\triangle$s BAD, BEC,

we have $BA : AD = BE : EC$; *VI. 4*

$\therefore$ EC is known.

Construct therefore the isosceles triangle ACE;

produce EA to B, making AB = the other given side;

and join BC.

(2) Let ABC (fig. to VI. A) be the triangle required. To obtain it, we know AB, AC, AD.

Through C draw $CE \parallel DA$, meeting BA at E.

Then, as in VI. A, $AE = AC$; $\therefore$ AE is known;

$\therefore$ BE is known.

Now from the similar $\triangle$s BAD, BEC,

we have $BA : AD = BE : EC$; *VI. 4*

$\therefore$ EC is known.

Construct therefore the isosceles triangle ACE;

produce AE to B, making AB = the other given side;

and join BC.

PROPOSITION C.

1. Because $AB \cdot AC = AD \cdot AE$,

$\therefore$ $AB : AD = AE : AC$.

Now in $\triangle$s ABD, AEC, since $\angle$s ABD, AEC, which a

opposite AD, AC, one pair of homologous sides, are equal, *III.* 21

$\therefore$ $\angle$s ADB, ACE opposite AB, AE, the other pair of homologous sides, are either equal or supplementary. *VI.* 7

But $\angle\ ACE$ is right; *III.* 31

$\therefore$ $\angle\ ADB$ is also right.

2. Let BC be the given base.

On BC describe a segment of a circle containing an angle equal to the given vertical angle, and complete the circle.

Then the diameter of this circle is known.

Since the rectangle contained by this diameter, and the altitude of the triangle = the rectangle contained by the sides, the altitude of the triangle can be found by the fourth deduction from II. 14.

The problem is then reduced to the third deduction from III. 33.

3. Let ABC (fig. to VI. C) be a triangle inscribed in a circle, and let AD, AE be drawn making $\angle\ BAD = \angle\ EAC$:

to prove $AB \cdot AC = AD \cdot AE$.

Because $\angle\ BAD = \angle\ EAC$,

and $\angle\ ABD = \angle\ AEC$; *III.* 21 or 22, *Cor.*

$\therefore$ $\triangle$s ABD, AEC are similar;

$\therefore$ $AB : AD = AE : AC$; *VI.* 4

$\therefore$ $AB \cdot AC = AD \cdot AE$.

4. If AD, AE coincide, AD bisects the vertical $\angle\ BAC$.

Now $AB \cdot AC = AD \cdot AE$,

$= ED \cdot AD + AD^2$, *II.* 3

$= BD \cdot DC + AD^2$; *III.* 35

$\therefore$ $AD^2 = AB \cdot AC - BD \cdot DC$.

If AD, AE do not coincide, but are in the same straight line, AD bisects the exterior vertical $\angle\ B'AC$.

Now $AB \cdot AC = AD \cdot AE$,

$= ED \cdot AD - AD^2$, *II.* 3

$= BD \cdot DC - AD^2$; *III.* 35, *Cor.*

$\therefore$ $AD^2 = BD \cdot DC - AB \cdot AC$.

If $\angle\ ADB$ is right, then $\angle\ ACE$ must be right,

and AE is a diameter of the circle;

and $AB \cdot AC = AD \cdot AE$.

5. Let AB (fig. to VI. C) be denoted by c, AC by b, AD by p_1, and AE by $2R$;

then $2\,R\,p_1 = bc$; $\therefore\ R = \dfrac{bc}{2\,p_1}$.

Similarly $R = \dfrac{ca}{2\,p_2} = \dfrac{ab}{2\,p_3}$.

Again $2\,R\,p_1 = bc$;

$\therefore\ 2\,R\,ap_1 = abc$;

$\therefore\ 2\,R \cdot 2\,\Delta = abc$;

$\therefore\ \Delta = \dfrac{abc}{4\,R}$.

6. Because the rectangles contained by the sides are equal to the rectangles contained by the circumscribed diameter and the perpendiculars on the third sides; and the circumscribed diameter is constant;

$\therefore$ the rectangles contained by the sides vary as, or are proportional to, the perpendiculars on the third sides.

7. From B draw $BG \perp AC$, and from C draw $CH \perp BD$.

Then $BA \cdot BC : CB \cdot CD = BG : CH$, by the preceding deduction,

$= BF : CF$, since the right-angled Δs BFG, CFH are similar.

Conversely: If $ABCD$ be a quadrilateral, whose diagonals intersect at F, and if $BA \cdot BC : CB \cdot CD = BF : CF$, then a circle may be circumscribed about $ABCD$.

Make the same construction as before.

Then $BF : CF = BG : CH$; *VI. 4*

$\therefore\ BA \cdot BC : CB \cdot CD = BG : CH$;

$\therefore\ BA : CD = BG : CH$;

$\therefore\ BA : BG = CD : CH$, by alternation.

Now in Δs ABG, DCH, since $\angle$s AGB, DHC, which are opposite BA, CD, one pair of homologous sides, are equal,

$\therefore\ \angle$s BAG, CDH, opposite BG, CH, the other pair of homologous sides, are either equal or supplementary. *VI. 7*

But they cannot be supplementary, since they are both acute;

$\therefore$ they must be equal;

$\therefore$ a circle may be circumscribed about $ABCD$. *III. 21*

8. The previous deduction proves

$AB \cdot AD : BA \cdot BC = AF : BF$, (1)

$BA \cdot BC : CB \cdot CD = BF : CF$, (2)

$CB \cdot CD : DC \cdot DA = CF : DF$. (3)

From (1) and (2) we obtain, by direct equality,

$AB \cdot AD : CB \cdot CD = AF : CF;$ (4)

from (2) and (3) we obtain, by direct equality,

$BA \cdot BC : DC \cdot DA = BF : DF.$ (5)

From (4) and (5) we obtain, by addition,

$AB \cdot AD + CB \cdot CD : CB \cdot CD = AC : CF,$

$BA \cdot BC + DC \cdot DA : DC \cdot DA = BD : DF.$

Now $CB \cdot CD : DC \cdot DA = CF : DF;$

$\therefore AB \cdot AD + CB \cdot CD : BA \cdot BC + DC \cdot DA = AC : BD.$

PROPOSITION D.

1. Let ABC be an equilateral triangle inscribed in a circle, and let D be any point in the arc cut off by AC: to prove $BD = AD + CD$.

Because $AB \cdot CD + AD \cdot BC = AC \cdot BD;$ *VI.* D

$\therefore \quad AB \cdot CD + AD \cdot AB = AB \cdot BD;$

$\therefore \quad CD + AD = BD.$

[See the nineteenth deduction of Book III.]

2. Let $ABCD$ be a quadrilateral which cannot be inscribed in a circle, AC, BD its diagonals: to prove $AC \cdot BD$ less than $AB \cdot CD + AD \cdot BC$.

Make $\angle BAE = \angle DAC$, and $\angle ABE = \angle ACD$, and let AE, BE intersect at E.

Since C is not on the $\bigcirc^{ce}$ of the circle passing through A, B, D,

$\therefore \angle ACD$ is not $= \angle ABD;$

$\therefore \angle ABE$ is not $= \angle ABD;$

$\therefore$ E does not fall on the diagonal BD. Join ED.

In $\triangle$s ABE, ACD, $\left\{ \begin{array}{l} \angle BAE = \angle CAD \\ \angle ABE = \angle ACD; \end{array} \right.$

$\therefore AB : BE = AC : CD;$ *VI.* 4

$\therefore AB \cdot CD = AC \cdot BE.$ *VI.* 16

Because $\angle BAE = \angle CAD$, $\therefore \angle BAC = \angle DAE;$

and because $\triangle$s ABE, ACD are similar,

$\therefore AB : AC = AE : AD;$ *VI.* 4

$\therefore \triangle$s ABC, AED are similar, *VI.* 6

and $BC : AC = ED : AD;$

$\therefore AD \cdot BC = AC \cdot ED.$ *VI.* 16

Hence $AB \cdot CD + AD \cdot BC = AC \cdot BE + AC \cdot ED$
$= AC \cdot (BE + ED)$.

But BD is less than $BE + ED$;

$\therefore$ $AC \cdot BD$ is less than $AC \cdot (BE + ED)$,

$\therefore$ $AC \cdot BD$ is less than $AB \cdot CD + AD \cdot BC$.

3. If the rectangle contained by the diagonals of a quadrilateral be equal to the sum of the two rectangles contained by the opposite sides, a circle can be circumscribed about the quadrilateral.

This is proved by the method of *reductio ad absurdum*.

4. Join FA, and draw $CG \parallel FA$, and meeting AB in G.

Then $\angle CGB = \angle FAD$; *I.* 29

and $\angle AFD = \angle AED$, *III.* 21

$= \angle CBG$; *III.* 22, *Cor.*

$\therefore$ $\triangle$s FAD, BGC are similar,

and $FD : FA = BC : BG$; *VI.* 4

$\therefore$ $FD \cdot BG = FA \cdot BC$. *VI.* 16

Because CG is $\parallel FA$,

$\therefore$ $\angle AGC =$ supplement of $\angle FAD$, *I.* 29

$= \angle FED$; *III.* 22

and $\angle CAG = \angle DFE$; *III.* 21

$\therefore$ $\triangle$s FED, AGC are similar,

and $FD : FE = AC : AG$; *VI.* 4

$\therefore$ $FD \cdot AG = FE \cdot AC$. *VI.* 16

Hence $FD \cdot BG + FD \cdot AG = FA \cdot BC + FE \cdot AC$;

$\therefore$ $FD \cdot AB = FA \cdot BC + FE \cdot AC$; *II.* 1

$\therefore$ $FC \cdot AB + FD \cdot AB = FC \cdot AB + FA \cdot BC + FE \cdot AC$.

$\therefore$ $AB \cdot (FC + FD) = AC \cdot FB + FE \cdot AC$, *VI.* D

$= AC \cdot (FB + FE)$; *II.* 1

$\therefore$ $FE + FB : FC + FD = AB : AC$.

DEDUCTIONS.

1. If possible, let $AC : CB = AD : DB$.

   Then $AB : CB = AB : DB$, by addition, or subtraction;

   $\therefore$ $CB = DB$, which is not the case.

2. If P be a point (fig. to III. 7) inside the circle, the greatest and least straight lines that can be drawn to the $\bigcirc^{ce}$ are PA and PD; and the geometric mean between them is the straight line drawn from P to the $\bigcirc^{ce}$ $\perp$ AD. *VI.* 13

   If P be a point (fig. to III. 8) outside the circle, the greatest and least straight lines that can be drawn to the $\bigcirc^{ce}$ are PA and PD; and the geometric mean between them is the tangent from P to the $\bigcirc^{ce}$.

   For the square on the tangent $= PA \cdot PD$. *III.* 36

3. Because $AB \cdot BC = AC^2$,

   $\therefore$ $AB : AC = AC : BC$; *VI.* 17

   $\therefore$ $\triangle\ ABD : \triangle\ ACD = \triangle\ ACD : \triangle\ BCD$. *VI.* 1

4. Take any straight line AC (fig. to VI. 13), and on it describe the semicircle ADC. Divide AC internally at B, so that $AC \cdot CB = AB^2$; *II.* 11

   Draw $BD \perp AC$, and join AD, CD.

   Then $AB = CD$, by the first deduction from VI. 30.

   But $AC : AD = AD : AB$; *VI.* 8, *Cor.*

   $\therefore$ $AC : AD = AD : CD$.

5. Let the circles DCF, ECG, whose centres are A and B, and whose diameters are DAF, EBG, touch each other externally at C, and let DE be a common tangent to the circles: to prove DE a geometric mean between DF and EG.

   Join AB, which passes through C, DC, EC, FC, GC.

   Then DF is $\parallel$ EG; *III.* 18, *I.* 28

   $\therefore$ $\angle\ FAC = \angle\ EBC$. *I.* 29

   Now $FA : AC = EB : BC$, since each is a ratio of equality;

   $\therefore$ $\angle\ ACF = \angle\ BCE$; *VI.* 6

   $\therefore$ EC and CF are in the same straight line.

   Similarly DC and CG are in the same straight line.

But since $\angle\ DCF$ is right, *III.* 31
$\therefore\ \angle\ FDE$ is similar to $\triangle\ DCE$; *VI.* 8
and since $\angle\ ECG$ is right, *III.* 31
$\therefore\ \triangle\ GED$ is similar to $\triangle\ ECD$; *VI.* 8
$\therefore\ \triangle\ FDE$ is similar to $\triangle\ DEG$;
$\therefore\ FD : DE = DE : EG$.

6. The proof will be given for a regular hexagon and the inscribed and circumscribed equilateral triangles.

Let $ABCDEF$ (fig. to IV. 15) be a regular hexagon inscribed in a circle.
Join AC, CE, EA forming an inscribed equilateral triangle; at A, C, E draw tangents to the circle, forming by their intersections a circumscribed equilateral $\triangle\ GHK$, G being opposite to A, H to C, and K to E.
Join OG, which may be proved to pass through D, and let OG meet CE at L; join OE.

Then $\triangle\ OLE = \frac{1}{6}\ \triangle\ ACE$,
$\triangle\ ODE = \frac{1}{6}\ ABCDEF$,
$\triangle\ OGE = \frac{1}{6}\ \triangle\ GHK$.
Now $OL : OE = OE : OG$; *VI.* 8, *Cor.*
$\therefore\ OL : OD = OD : OG$;
$\therefore\ \triangle\ OLE : \triangle\ ODE = \triangle\ ODE : \triangle\ OGE$; *VI.* 1
$\therefore\ \triangle\ ACE : ABCDEF = ABCDEF : \triangle\ GHK$.

7. Because $\triangle\ DAE$ is isosceles,
$\therefore\ AD^2 - BD^2 = AB \cdot BE$, by first deduction of Book II.,
$= AB \cdot (AB - BC)$,
$= AB^2 - AB \cdot BC$;
$\therefore\ BD^2 = AB \cdot BC$.

8. Let AC (fig. to VI. 13) be the sum of the extremes.

From C draw $CE \perp AC$, and $=$ the given mean;
through E draw $ED \parallel AC$, and cutting the semicircle at D.
From D draw $DB \perp AC$.

AB and BC are the extremes.

The proof follows from VI. 13.

9. Let AC be the difference of the extremes.

On AC as diameter describe a circle, in which take any point D, and draw a tangent $DE =$ the given mean.
Join E with the centre O, and produce EO to meet the $\bigcirc^{ce}$ at G.
With centre O and radius OE, describe another circle cutting AC produced at B. AB and BC are the extremes.

For BC may be proved $= EF$, and $AB = EG$;

$\therefore \qquad AB \cdot BC = GE \cdot EF,$

$= DE^2.$ *III.* 36

10. Let AB be one extreme, BC the sum of the mean and the other extreme.

Place AB and BC in the same straight line, and on AC describe a segment of a circle AEC, containing an angle equal to the exterior angle of an equilateral triangle.

At B make $\angle\ ABE =$ the angle in the segment, and from E draw ED, making $\angle\ EDC =$ the angle in the segment.

BD and DC are the mean and the other extreme.

Because $\angle$s ABE, CDE are each $=$ the exterior angle of an equilateral triangle;

$\therefore\ \triangle\ EBD$ is equilateral.

Now $\triangle$s ABE, EDC are each similar to $\triangle\ AEC$;

$\therefore\ \triangle\ ABE$ is similar to $\triangle\ EDC$;

$\therefore\ AB : BE = ED : DC;$

$\therefore\ AB : BD = BD : DC.$

11. Let the 1st term, and the difference between the 2nd and 3rd, be given: to find the 2nd and 3rd.

(*a*) Suppose the 1st to be the greatest of the three terms.

Since 1st : 2nd = 2nd : 3rd,

$\therefore$ 1st : 2nd = 1st − 2nd : 2nd − 3rd;

$\therefore$ 1st · (2nd − 3rd) = 2nd · (1st − 2nd).

Now since 1st and (2nd − 3rd) are given, the rectangle contained by them is given;

$\therefore$ the rectangle contained by the 2nd and (1st − 2nd) is given.

Also since 2nd + (1st − 2nd) = 1st, the sum of the sides which contain the rectangle is given;

hence, the area of the rectangle and the sum of the sides which contain it being given, the sides themselves may be found.

Thus the 2nd term of the series is obtained, and since the (2nd − 3rd) is known, the 3rd is readily found.

(*b*) Suppose the 1st to be the least of the three terms.

Since 1st : 2nd = 2nd : 3rd,

$\therefore$ 1st : 2nd = 2nd − 1st : 3rd − 2nd;

$\therefore$ 1st · (3rd − 2nd) = 2nd · (2nd − 1st).

Now since 1st and (3rd − 2nd) are given, the rectangle contained by them is given;

$\therefore$ the rectangle contained by 2nd and (2nd − 1st) is given. Also since 2nd − (2nd − 1st) = 1st, the difference of the sides which contain the rectangle is given;
hence, the area of the rectangle and the difference of the sides which contain it being given, the sides themselves may be found.
Thus the 2nd term of the series is obtained, and since the (3rd − 2nd) is known, the 3rd is readily found.

For the geometrical construction of (a), use VI. 28, and for that of (b), use VI. 29.

12. This problem comprehends fifteen cases, the data of which are

(1) Sum, and difference.
(2) Sum of squares, and difference of squares.
(3) Sum, and sum of squares.
(4) Difference, and sum of squares.
(5) Sum, and difference of squares.
(6) Difference, and difference of squares.
(7) Sum, and rectangle.
(8) Difference, and rectangle.
(9) Sum of squares, and rectangle.
(10) Difference of squares, and rectangle.
(11) Sum, and ratio.
(12) Difference, and ratio.
(13) Sum of squares, and ratio.
(14) Difference of squares, and ratio.
(15) Rectangle, and ratio.

(1) See the sixth and seventh deductions from I. 3.

(2) Let AB^2 be the given sum of squares, CD^2 the given difference of squares.

Find a square $= \frac{1}{2}(AB^2 + CD^2)$, and another square $= \frac{1}{2}(AB^2 - CD^2)$; the sides of these squares will be the required straight lines.

(3) Let AB^2 be the given sum of squares, and M the given sum.

On AB as base describe a segment of a circle containing an angle = half a right angle.
With A as centre and M as radius, cut the arc of the segment at D.
Join AD, and let it cut at C the arc of a semicircle described on AB as diameter.

AC and CB are the required straight lines.

(4) Let AB^2 be the given sum of squares, and N the given difference.

On AB as base describe a segment of a circle containing an angle $=$ a right angle and a half.

With A as centre and N as radius, cut the arc of the segment at D.

Join AD, and produce it to cut at C the arc of a semicircle described on AB as diameter.

AC and CB are the required straight lines.

(5) Let AB^2 be the given difference of squares, and M the given sum.

Describe a rectangle $= AB^2$, and having one of its sides $= M$, by the fourth deduction from II. 14.

The other side of the rectangle will be the difference of the two straight lines. *II.* 5, *Cor.*

The problem is now reduced to (1).

(6) Let AB^2 be the given difference of squares. and N the given difference.

Describe a rectangle $= AB^2$, and having one of its sides $= N$, by the fourth deduction from II. 14.

The other side of the rectangle will be the sum of the two straight lines. *II.* 5, *Cor.*

The problem is now reduced to (1).

(7) This case is solved in VI. 28.

(8) This case is solved in VI. 29.

(9) Let AC^2 (fig. to VI. 13) be the given sum of squares.

Convert the given rectangle into a square; *II.* 14

and find, by the fourth deduction from II. 14, a straight line E such that the rectangle contained by AC and E may be equal to this square.

Divide AC internally at B so that $AB \cdot BC = E^2$, by the second deduction from II. 14.

On AC as diameter describe a semicircle, and draw $BD \perp AC$.

AD, CD are the required straight lines.

For $\quad BD^2 = AB \cdot BC,$ *VI.* 8, 17

$\qquad\qquad = E^2;$

$\therefore \quad BD = E.$

But $AD \cdot CD = AC \cdot BD,$ *VI.* 8, 16

$\qquad\qquad = AC \cdot E;$

and $AD^2 + CD^2 = AC^2.$ *III.* 31, *I.* 47

(10) Let AB^2 (fig. to VI. 13) be the given difference of squares.

Convert the given rectangle into a square; *II.* 14
and find, by the fourth deduction from II. 14, a straight line E such that the rectangle contained by AB and E may be equal to this square.
Divide AB externally at C so that $AC \cdot CB = E^2$, by the third deduction from II. 14.
On AC as diameter describe a semicircle, and draw $BD \perp AC$.
AD, BD are the required straight lines.

Join CD.

Then $CD^2 = AC \cdot CB$, *VI.* 8, 17
$= E^2$;
$\therefore$ $CD = E$.
But $AD \cdot BD = AB \cdot CD$, *VI.* 8, 16
$= AB \cdot E$;
and $AD^2 - BD^2 = AB^2$. *I.* 47, *Cor.*

(11) (12) These cases are solved in VI. 10.

(13) Let $AB : AC$ (AB and AC are placed as in VI. 13) be the given ratio, and let D^2 be the given sum of squares.

At A draw $AE \perp AC$ and $= AC$; join BE, and from it produced if necessary cut off $BF = D$. Through F draw $FG \parallel EA$, meeting BA or BA produced at G.
GB, GF are the required straight lines.

For $GB^2 + GF^2 = BF^2$, *I.* 47
$= D^2$;
and $GB : GF = AB : AE$, *VI.* 4
$= AB : AC$.

(14) Let $AB : AC$ (AB and AC are placed as in VI. 13) be the given ratio, and let D^2 be the given difference of squares.

At B draw $BE \perp AC$, and make $AE = AC$; join AE, and from BE produced if necessary cut off $BF = D$. Through F draw $FG \parallel EA$, meeting BA or BA produced at G.
GB, GF are the required straight lines.

For $GF^2 - GB^2 = BF^2$, *I.* 47, *Cor.*
$= D^2$;
and $GB : GF = AB : AE$, *VI.* 4
$= AB : AC$.

(15) Let $AB \cdot BC$ be the given rectangle, $K : L$ the given ratio.

Place AB, BC so as to contain any angle ABC, and from BC produced if necessary cut off BD such that
$K : L = AB : BD$. *VI.* 12
Join AD, and from BD cut off BE a mean proportional between BC and BD. *VI.* 13
Through E draw $EF \parallel AD$, meeting AB at F.
BF, BE are the required straight lines.

For $\quad BF : BA = BE : BD$, *VI.* 4
and $\quad BE : BD = BE : BD$;
$\therefore BF \cdot BE : AB \cdot BD = BE^2 : BD^2$,
$= BC : BD$, *VI.* 20, *Cor.*
$= AB \cdot BC : AB \cdot BD$; *VI.* 1
$\therefore \quad BF \cdot BE = AB \cdot BC$,
$=$ the given rectangle.
Now $\quad BF : BE = BA : BD$, *VI.* 4
$= K : L$.

[Some of these solutions have been taken from *Geometry*, by Pierce Morton, in the Library of Useful Knowledge, pp. 123–4.]

13. Let ABC, DCE be two triangles having $\angle ACB$ supplementary to $\angle DCE$, and $\angle BAC = \angle EDC$;
to prove $AB : BC = DE : EC$.

Place the triangles so that the supplementary angles may be contiguous, and so that BC and CE, which are opposite the equal angles, may be in the same straight line. Through A draw $AF \parallel DE$, and meeting CE or CE produced at F.

Then $\angle BAC = \angle EDC = \angle FAC$; *I.* 29
$\therefore \quad AB : AF = BC : FC$; *VI.* 3
$\therefore \quad AB : BC = AF : FC$, by alternation,
$= DE : EC$. *VI.* 4

14. Let AB be the given straight line, $K : L$ the given ratio.
Find M a mean proportional between K and L; *VI.* 13
divide AB internally at C in the ratio of $K : M$. *VI.* 10
Then $AC : CB = K : M$;
$\therefore \quad AC^2 : CB^2 = K^2 : M^2$; *VI.* 22
$= K : L$. *VI.* 20, *Cor.*

15. Let M be the given polygon, $K : L$ the given ratio.
Find AB the side of a square $= M$; *II.* 14
and from AB or AB produced cut off AC such that
$K : L = AB : AC$. *VI.* 12
Find a square $D =$ rectangle $AB \cdot AC$. *II.* 14

Then $K : L = AB : AC,$
$= AB^2 : AB \cdot AC,$ *VI.* 1
$= AB^2 : D.$

16. Let ABC be the given triangle, $K : L$ the given ratio.
Find M a mean proportional between K and L; *VI.* 13
from AB cut off AD such that $K : M = AB : AD$; *VI.* 12
through D draw $DE \parallel BC$, meeting AC at E.
Then $\triangle ABC : \triangle ADE$ = duplicate of $AB : AD$, *VI.* 19
$= AB^2 : AD^2,$
$= K^2 : M^2,$
$= K : L.$

17. Let BAC be the given angle, $K : L$ the given ratio, and M^2 the given space.
Make $AB = K$, $AC = L$, and join BC. Find N such that $N^2 = \triangle ABC$. *II.* 14
From AB or AB produced cut off AD such that $N : M = AB : AD$, and through D draw $DE \parallel BC$, meeting AC or AC produced at E.
Then $N^2 : M^2 = AB^2 : AD^2,$ *VI.* 22
= duplicate of $AB : AD$,
$= \triangle ABC : \triangle ADE.$ *VI.* 19
But $N^2 = \triangle ABC;$
$\therefore$ $M^2 = \triangle ADE;$
and $AD : AE = AB : AC,$ *VI.* 4
$= K : L.$

18. From F draw $FH \parallel DA$, meeting AC at H, and from G draw $GK \parallel EB$, meeting BC at K, and join HK. Join also AK, BH.
Then $HFGK$ is a rectangle, similar and similarly situated to $ADEB$, by the ninth deduction from I. 20;
$\therefore$ $HK^2 : HF^2 = AB^2 : AD^2 = 2 : 1.$
Now $AG^2 + BF^2 = AK^2 - KG^2 + BH^2 - HF^2$, *I.* 47, *Cor.*
$= AC^2 + CK^2 - KG^2 + BC^2 + CH^2 - HF^2,$
$= (AC^2 + BC^2) + (CK^2 + CH^2) - 2\,HF^2,$
$= AB^2 + HK^2 - 2\,HF^2,$
$= AB^2.$
[This solution is due to W. J. Macdonald.]

19. Let $ADBC$ be a circle, and let the chords AB, CD intersect at E. If O be the centre of the circle, the chord which passes through E and is there bisected is $\perp OE$; let HK be this chord. Let AD, BC cut HK at F, G; to prove $EF = EG$.

Through G draw $ML \parallel AD$, and meeting EB, EC, produced when necessary, at L, M.

Then $\angle\ BLG = \angle\ DAB$, — I. 29

$= \angle\ MCG$; — III. 21

$\therefore$ $\triangle$s BLG, MCG are mutually equiangular;

$\therefore$ $BG : GL = MG : GC$; — VI. 4

$\therefore$ $MG \cdot GL = BG \cdot GC$, — VI. 16

$= HG \cdot GK$, — III. 35

$= EK^2 - EG^2$. — II. 5

And $AF \cdot DF = HF \cdot FK$, — III. 35

$= EH^2 - EF^2$, — II. 5

$= EK^2 - EF^2$.

Because $\triangle$s AEF, LEG are mutually equiangular,

$\therefore$ $AF : FE = LG : GE$; — VI. 4

because $\triangle$s DEF, MEG are mutually equiangular,

$\therefore$ $DF : FE = MG : GE$; — VI. 4

$\therefore$ $AF \cdot DF : FE^2 = LG \cdot MG : GE^2$;

$\therefore$ $EH^2 - EF^2 : FE^2 = EK^2 - EG^2 : GE^2$;

$\therefore$ $EH^2 : FE^2 = EK^2 : GE^2$, by addition.

But $EH^2 = EK^2$; $\therefore$ $FE^2 = GE^2$, and $FE = GE$.

20. If two chords AB, CD intersect each other at a point E outside a circle, the straight lines AD, BC cut off equal segments from the straight line which passes through E and is perpendicular to the diameter through E.

The construction and proof are similar to those of the preceding deduction, with only a few slight modifications.

[A more general theorem, of which the nineteenth and twentieth deductions are particular cases, will be found in the *Proceedings of the Edinburgh Mathematical Society*, vol. iii., session 1884–85, p. 38.]

21. Because $\triangle$s AFI, AF_1I_1 (fig. on p. 251 of *Euclid*) are mutually equiangular,

$\therefore$ $AF : IF = AF_1 : I_1F_1$, — VI. 4

But $AF_1 : AF_1 = IF : IF$;

$\therefore$ $AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : IF \cdot I_1F_1$.

Because $\triangle$s IBF, I_1BF_1 are mutually equiangular,

$\therefore$ $BF : IF = I_1F_1 : BF_1$; — VI. 4

$\therefore$ $IF \cdot I_1F_1 = BF \cdot BF_1$. — VI. 16

Hence $AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : BF \cdot BF_1$;

$\therefore$ $s(s-a) : \triangle = \triangle : (s-b)(s-c)$,

by (1) and (2) of the nineteenth deduction, and the twenty-fifth of Book IV.

22. Because $\triangle$s AF_1I_1, AFI are mutually equiangular,

$\therefore\ AF_1 : AF = I_1F_1 : IF,$ *VI.* 4

$= I_1F_1 \cdot IF : IF^2,$

$= BF \cdot BF_1 : IF^2$, by the previous deduction;

$\therefore\ s : s - a = (s - b)(s - c) : r^2.$

$AF_1 : AF = I_1F_1 : IF,$

$= I_1F_1^2 : I_1F_1 \cdot IF,$

$= I_1F_1^2 : BF \cdot BF_1$, by the previous deduction;

$\therefore\ s : s - a = r_1^2 : (s - b)(s - c).$

23. From the twenty-first deduction there results

$\triangle^2 = s(s - a)(s - b)(s - c).$

From the twenty-fifth deduction of Book IV. there results

$rr_1r_2r_3 \cdot s(s - a)(s - b)(s - c) = \triangle^4;$

$\therefore\ rr_1r_2r_3 \cdot \triangle^2 = \triangle^4;$

$\therefore\ rr_1r_2r_3 = \triangle^2.$

24. Because $\triangle$s I_1AF_1, I_2AF_2 are mutually equiangular,

$\therefore\ I_1F_1 : AF_1 = AF_2 : I_2F_2;$ *VI.* 4

$\therefore\ AF_1 \cdot AF_2 = I_1F_1 \cdot I_2F_2$, or $s(s - c) = r_1r_2$.

Similarly $s(s - a) = r_2r_3$, and $s(s - b) = r_3r_1;$

$\therefore\ r_1r_2 + r_2r_3 + r_3r_1 = s(s - c + s - a + s - b),$

$= s\{3s - (a + b + c)\} = s^2.$

25. Because $\triangle$s I_2D_2B, I_3D_3B are mutually equiangular,

$\therefore\ I_2D_2 : BD_2 = BD_3 : I_3D_3;$ *VI.* 4

$\therefore\ r_2 : s = s - a : r_3;$

$\therefore\ s^2 - sa = r_2r_3.$

But $s^2 = r_1r_2 + r_2r_3 + r_3r_1$, by the previous deduction.

$\therefore$ by subtraction, $sa = r_1r_2 + r_3r_1 = r_1(r_2 + r_3).$

Similarly $sb = r_2(r_3 + r_1),$

and $sc = r_3(r_1 + r_2).$

26. Because $\triangle$s IBD, I_3BD_3 are mutually equiangular,

$\therefore\ BD : ID = I_3D_3 : BD_3;$ *VI.* 4

$\therefore\ ID \cdot I_3D_3 = BD \cdot BD_3,$ *VI.* 16

$= (s - a) \cdot (s - b),$

$\therefore\ rr_3 = AF \cdot FB.$

Similarly $rr_1 = BD \cdot DC,$

and $rr_2 = CE \cdot EA;$

$\therefore\ r(r_1 + r_2 + r_3) = AF \cdot FB + BD \cdot DC + CE \cdot EA.$

27. About $\triangle$ ABC circumscribe a circle whose centre is S; join

BS, and produce it to meet the $\bigcirc^{ce}$ at B'; join CB', and from S draw $SH \perp BC$.

Because A, Z, O, Y are concyclic,

$\therefore\ \angle\ ZAO = \triangle\ OYZ$; *III.* 21

$\therefore\ \triangle$s ABO, YBZ are mutually equiangular;

$\therefore\quad AO : YZ = AB : YB$. *VI.* 4

Because $\angle\ BAY = \angle\ BB'C$, *III.* 21

and $\angle\ BYA = \angle\ BCB'$; *III.* 31

$\therefore\ \triangle$s BAY, $BB'C$ are mutually equiangular;

$\therefore\quad AB : YB = B'B : CB$. *VI.* 4

Hence $\quad AO : YZ = B'B : CB$,

and $\quad AO \cdot BC = B'B \cdot YZ$; *VI.* 16

$\therefore\quad 2\,SH \cdot BC = 2\,R \cdot YZ$. *App. I.* 5, *Cor.*

Similarly $2\,SK \cdot CA = 2\,R \cdot ZX$,

and $\quad 2\,SL \cdot AB = 2\,R \cdot XY$;

$$\begin{aligned}\therefore\ R\,(XY + YZ + ZX) &= SH \cdot BC + SK \cdot CA + SL \cdot AB,\\ &= 2\,\triangle\ SBC + 2\,\triangle\ SCA\\ &\qquad + 2\,\triangle\ SAB,\\ &= 2\,\triangle\ ABC;\end{aligned}$$

$\therefore\ \frac{1}{2} R \cdot (XY + YZ + ZX) = \triangle\ ABC$.

28. For $r \cdot 2s = 2\,\triangle\ ABC$, by twenty-fifth deduction of Book IV.,
$= R \cdot (XY + YZ + ZX)$, by the previous deduction;
$\therefore\ 2s : XY + YZ + ZX = R : r$.

29. For $2\,\triangle\ XYZ = \rho\,(XY + YZ + ZX)$, by the twenty-fifth deduction of Book IV.;
and $2\,\triangle\ ABC = R \cdot (XY + YZ + ZX)$, by the twenty-seventh deduction;

$$\begin{aligned}\therefore\ ABC : XYZ &= R\,(XY + YZ + ZX) : \rho\,(XY + YZ + ZX),\\ &= R : \rho.\end{aligned}$$

30. Draw $OE' \perp ZX$.

Then OE' is a radius of the inscribed circle of $\triangle\ XYZ$, by the twenty-first deduction of Book IV.;
and $\triangle$s $B'BC$, ABY are mutually equiangular, by the twenty-seventh deduction;

$\therefore\ \angle\ B'BC = \angle\ ABY = \angle\ ZBO$,
$= \angle\ ZXO$, since Z, B, X, O are concyclic;

and $\angle\ BCB' = \angle\ OE'X$. *III.* 31

Hence $\triangle$s $B'BC$, OXE' are mutually equiangular;

$\therefore\quad BB' : CB' = XO : E'O$, *VI.* 4

$\therefore\quad BB' \cdot E'O = CB' : OX$,
$= 2\,SH \cdot OX$, *App. I.* 1

$= AO \cdot OX;$ *App. I.* 5, *Cor.*

$\therefore \quad 2R\rho = AO \cdot OX.$

Similarly $2R\rho = BO \cdot OY = CO \cdot OZ.$

31. See the figures to the four preceding deductions.

Because $\angle BAX = \angle BCZ = \angle OCX$,

$\therefore$ $\triangle$s BXA, OXC are mutually equiangular;

$\therefore \quad BX : XA = OX : XC;$ *VI.* 4

$\therefore \quad BX \cdot XC = OX \cdot XA,$ *VI.* 16

$= AO \cdot OX + OX^2.$ *II.* 3

Now $BX \cdot XC + HX^2 = HB^2 = \frac{1}{4}BC^2;$ *II.* 5

$\therefore \quad AO \cdot OX + OX^2 + HX^2 = \frac{1}{4}BC^2.$

Again, $AH^2 = (AO + OX)^2 + HX^2,$ *I.* 47

$= AO^2 + OX^2 + 2AO \cdot OX + HX^2,$

$= \frac{1}{4}BC^2 + AO^2 + AO \cdot OX;$

But $AB^2 + AC^2 = \frac{1}{2}BC^2 + 2AH^2,$ *App. II.* 1

$= BC^2 + 2AO^2 + 2AO \cdot OX;$

$\therefore AB^2 + BC^2 + CA^2 = 2BC^2 + 2AO^2 + 2AO \cdot OX.$

Lastly $BC^2 + AO^2 = BC^2 + 4SH^2,$ *App. I.* 5, *Cor.*

$= BC^2 + B'C^2,$ *App. I.* 1

$= BB'^2,$

$= 4R^2;$

and $AO \cdot OX = 2R\rho$, by the preceding deduction;

$\therefore AB^2 + BC^2 + CA^2 = 8R^2 + 4R\rho.$

32. Let ABC be a triangle circumscribed by a circle whose centre is S, I the inscribed centre, and I_1, the first escribed centre. $H'U$ (as in the fig. to the twenty-ninth deduction of Book IV.) is the diameter of the circumscribed circle bisecting BC; AI_1 is joined, passing through I and U; IE, I_1E_1 are drawn $\perp AC$, and CH', CU, SA, SI, SI_1 are joined.

Because $\triangle$s EAI, $CH'U$ are mutually equiangular, being right-angled, and having $\angle EAI = \angle CH'U;$ *III.* 21

$\therefore \quad AI : IE = H'U : UC;$ *VI.* 4

$\therefore H'U \cdot IE = AI \cdot UC,$ *VI.* 16

$= AI \cdot UI$, by the twenty-second deduction of Book IV.,

$= SA^2 - SI^2$, by the first deduction of Book II.;

$\therefore \quad 2Rr = R^2 - SI^2;$

$\therefore \quad SI^2 = R^2 - 2Rr.$

33. Because $\triangle$s E_1AI_1, $CH'U$ are mutually equiangular, being right-angled, and having $\angle E_1AI_1 = \angle CH'U;$ *III.* 21

$\therefore AI_1 : I_1E_1 = H'U : UC,$ *VI.* 4

$\therefore\ H'U \cdot I_1E_1 = AI_1 \cdot UC,$ *VI.* 16

$= AI_1 \cdot UI_1$, by the twenty-second deduction of Book IV.,

$= SI_1^2 - SA^2$, by first deduction of Book II.;

$\therefore\quad 2\,Rr_1 = SI_1^2 - R^2;$

$\therefore\quad SI_1^2 = R^2 + 2\,Rr_1.$

Similarly $SI_2^2 = R^2 + 2\,Rr_2$,

and $SI_3^2 = R^2 + 2\,Rr_3$.

34. For $SI^2 + SI_1^2 + SI_2^2 + SI_3^2$,

$= R^2 - 2\,Rr + R^2 + 2\,Rr_1 + R^2 + 2\,Rr_2 + R^2 + 2\,Rr_3,$

$= 4\,R^2 + 2\,R\,(r_1 + r_2 + r_3 - r),$

$= 4\,R^2 + 2\,R \cdot 4\,R$, by the twenty-ninth deduction of Book IV.,

$= 12\,R^2.$

35. See figure and demonstration to the twenty-ninth deduction of Book IV.

$BC^2 = 4\,BH^2 = 4\,BH \cdot HC = 4\,HH' \cdot HU;$ *III.* 35

$\therefore\quad a^2 = (r_2 + r_3)(r_1 - r).$

Similarly $b^2 = (r_3 + r_1)(r_2 - r)$,

and $c^2 = (r_1 + r_2)(r_3 - r);$

$\therefore\ a^2 + b^2 + c^2 = 2\,(r_1r_2 + r_2r_3 + r_3r_1) - 2\,r\,(r_1 + r_2 + r_3),$

$= 2\,(r_1r_2 + r_2r_3 + r_3r_1) - 2\,r\,(4\,R + r);$

$\therefore\ 2\,(r_1r_2 + r_2r_3 + r_3r_1) = a^2 + b^2 + c^2 + 2\,r\,(4\,R + r).$

Now since $r_1 + r_2 + r_3 = 4\,R + r$,

$\therefore\ r_1^2 + r_2^2 + r_3^2 + 2\,(r_1r_2 + r_2r_3 + r_3r_1) = 16\,R^2 + 8\,Rr + r^2;$

$\therefore\ r_1^2 + r_2^2 + r_3^2 + a^2 + b^2 + c^2 + 8\,Rr + 2\,r^2 = 16\,R^2 + 8\,Rr + r^2;$

$\therefore\ a^2 + b^2 + c^2 + r^2 + r_1^2 + r_2^2 + r_3^2 = 16\,R^2.$

36. See figure on p. 251 of *Euclid.*

Through I draw to I_1E_1 a parallel to EE_1, which is equal to a;

then $II_1^2 = a^2 + (r_1 - r)^2.$

Similarly $II_2^2 = b^2 + (r_2 - r)^2$,

and $II_3^2 = c^2 + (r_3 - r)^2.$

Again, through I_1 draw to I_2E_2 produced a parallel to E_1E_2, which is equal to c;

then $I_1I_2^2 = c^2 + (r_1 + r_2)^2.$

Similarly $I_2I_3^2 = a^2 + (r_2 + r_3)^2$,

and $I_3I_1^2 = b^2 + (r_3 + r_1)^2.$

Hence $II_1^2 + II_2^2 + II_3^2 + I_1I_2^2 + I_2I_3^2 + I_3I_1^2$,

$$= 2(a^2 + b^2 + c^2) + (r_1 - r)^2 + (r_2 - r)^2 + (r_3 - r)^2 + (r_1 + r_2)^2 + (r_2 + r_3)^2 + (r_3 + r_1)^2,$$
$$= 2(a^2 + b^2 + c^2) + 3(r^2 + r_1^2 + r_2^2 + r_3^2) + 2(r_1r_2 + r_2r_3 + r_3r_1) - 2r(r_1 + r_2 + r_3),$$
$$= 2(a^2 + b^2 + c^2) + 3(r^2 + r_1^2 + r_2^2 + r_3^2) + a^2 + b^2 + c^2,$$
$$= 3(a^2 + b^2 + c^2 + r^2 + r_1^2 + r_2^2 + r_3^2),$$
$$= 48 R^2.$$

37. Let ABC be a triangle, H the middle point of BC, AX the perpendicular from A on BC; AN, AN' the bisectors of the interior and exterior vertical angles at A; D, D_1, D_2, D_3 the points of contact of the inscribed and escribed circles with BC. [For convenience suppose AB greater than AC. The D points need not be marked on the figure.]

(1) From C draw $CM \perp AN$, and let CM produced meet AB at C'; join MH, MX.

Then $AC' = AC$, and $MC' = MC$; *I.* 26

$\therefore BC' = AB - AC$, and $HM = \frac{1}{2}(AB - AC)$. *App. I.* 1

Now $HD = HD_1 = \frac{1}{2}(AB - AC)$, by (15) and (5) of the nineteenth deduction of Book IV.;

$\therefore HM = HD = HD_1$.

Because $\angle$s AMC, AXC are right,

$\therefore$ the points A, M, X, C are concyclic;

$\therefore \angle HXM = \angle MAC$, *III.* 22, *Cor.*

$= \frac{1}{2} \angle BAC$.

But since HM is $\parallel BC'$, *App. I.* 1

$\therefore \angle HMN = \angle BAN$, *I.* 29

$= \frac{1}{2} \angle BAC$;

$\therefore \angle HXM = \angle HMN$.

Hence $\triangle$s HXM, HMN are mutually equiangular;

$\therefore HX : HM = HM : HN$; *VI.* 4

$\therefore HX \cdot HN = HM^2 = HD^2 = HD_1^2$.

(2) From C draw $CM \perp AN'$, and let CM produced meet BA produced at C'; join MH, MX.

Then $AC' = AC$, and $MC' = MC$; *I.* 26

$\therefore BC' = AB + AC$, and $HM = \frac{1}{2}(AB + AC)$. *App. I.* 1

Now $HD_2 = HD_3 = \frac{1}{2}(AB + AC)$, by (15) and (4) of the nineteenth deduction of Book IV.;

$\therefore HM = HD_2 = HD_3$.

Because $\angle$s AMC, AXC are right,

$\therefore$ the points A, M, X, C are concyclic;

$\therefore \quad \angle HXM =$ supplement of $\angle MXC$,
$=$ supplement of $\angle MAC$, *III.* 21
$=$ supplement of $\frac{1}{2}(B + C)$.

But since HM is $\parallel BC'$, *App. I.* 1
$\therefore \quad \angle HMN' = \angle BAM$, *I.* 29
$=$ supplement of $\angle MAC'$,
$=$ supplement of $\frac{1}{2}(B + C)$;
$\therefore \quad \angle HXM = \angle HMN'$.

Hence $\triangle$s HXM, HMN' are mutually equiangular;
$\therefore \; HX : HM = HM : HN'$; *VI.* 4
$\therefore \; HX \cdot HN' = HM^2 = HD_2^2 = HD_3^2$.

38. (1) $HX \cdot ND = HX \cdot (HD - HN)$,
$= HX \cdot HD - HX \cdot HN$,
$= HX \cdot HD - HD^2$, by the previous deduction,
$= HD \cdot (HX - HD)$,
$= HD \cdot DX$.

(2) $HX \cdot ND_1 = HX \cdot (HD_1 + HN)$,
$= HX \cdot HD_1 + HX \cdot HN$,
$= HX \cdot HD_1 + HD_1^2$, by the previous deduction,
$= HD_1 \cdot (HX + HD_1)$,
$= HD_1 \cdot D_1X$.

39. (1) $HX \cdot N'D_2 = HX \cdot (HN' - HD_2)$,
$= HX \cdot HN' - HX \cdot HD_2$,
$= HD_2^2 - HX \cdot HD_2$, by the thirty-seventh deduction,
$= HD_2 \cdot (HD_2 - HX)$,
$= HD_2 \cdot D_2X$.

(2) $HX \cdot N'D_3 = HX \cdot (HN' + HD_3)$,
$= HX \cdot HN' + HX \cdot HD_3$,
$= HD_3^2 + HX \cdot HD_3$, by the thirty-seventh deduction,
$= HD_3 \cdot (HD_3 + HX)$,
$= HD_3 \cdot D_3X$.

40. (1) $HN \cdot NX = HN \cdot (HX - HN)$,
$= HX \cdot HN - HN^2$,
$= HD^2 - HN^2$, by the thirty-seventh deduction,
$= (HD - HN) \cdot (HD + HN)$,
$= DN \cdot ND_1$.

$$
\begin{aligned}
(2)\ HN' \cdot N'X &= HN' \cdot (HN' - HX), \\
&= HN'^2 - HX \cdot HN', \\
&= HN'^2 - HD_2^2, \text{ by the thirty-seventh deduction}, \\
&= (HN' - HD_2) \cdot (HN' + HD_2), \\
&= D_2N' \cdot N'D_3.
\end{aligned}
$$

41. Let BC be the given base.

On BC describe a segment of a circle BAC containing an angle = the given vertical angle, and complete the circle. Bisect the arc conjugate to BAC at E, and divide BC internally at D in the ratio of the sides. Join ED, and produce it to meet the arc BAC at A; and join AB, AC.

ABC is the required triangle.

For $\angle\ BAD = \angle\ CAD$; *III.* 27

$\therefore\quad BA : AC = BD : DC.$ *VI.* 3

42. On the given diameter describe a circle, and from it cut off a segment BAC containing an angle = the given vertical angle.

Bisect the arc conjugate to BAC at E, and continue the construction and proof as in the preceding deduction.

43. Take any straight line DE, and on it describe a segment of a circle containing an angle = the given vertical angle.

Bisect DE at H, and at H draw HF, making with DE an angle = the angle which the median makes with the base; let HF meet the arc of the segment at F, and join FD, FE. Produce HF if necessary till HA = the given median, and through A draw $AB \parallel FD$, and $AC \parallel FE$, meeting DE, produced if necessary, at B and C.

ABC is the required triangle.

For $\triangle\ BAC = \angle\ DFE$; *I.* 34, *Cor.*

and $HD : HB = HF : HA,$ *VI.* 4

$\qquad = HE : HC.$ *VI.* 4

But $HD = HE$; $\therefore\ HB = HC.$

44. Take any straight line DE, and on it describe a segment of a circle containing an angle = the given vertical angle.

Divide DE internally at F in the ratio of the segments of the base made by the perpendicular; from F draw $FA \perp DE$, and meeting the arc of the segment at A, and join AD, AE.

From AF, produced if necessary, cut off AX = the given

perpendicular; through X draw $BXC \parallel DE$, and meeting AD, AE, produced if necessary, at B and C.

ABC is the required triangle.

For $BX : XC = DF : FE$.

45. Let ABC be the required triangle.

(1) When the sum of the sides is given.

About $\triangle\ ABC$ circumscribe a circle; draw the diameter PQ, as in the figure to the twenty-fifth deduction of Book III., bisecting the base BC at H, and join AP. From P draw $PS \perp AB$, $AE \perp PQ$, and $AX \perp BC$; from AB cut off $AC' = AC$, join CC', and produce SH to meet AP in D.

Then $AS = \frac{1}{2}(AB + AC)$, $BS = \frac{1}{2}(AB - AC)$, and AP bisects $\angle\ BAC$, by the twenty-fifth deduction of Book III.

But $BC' = AB - AC$;

$\therefore$ S is the middle point of BC';

$\therefore$ SH is $\parallel C'C$, since H bisects BC; *App. I.* 1

$\therefore$ SD is $\perp AP$, since $C'C$ is $\perp AP$;

$\therefore$ $PA \cdot PD = PS^2$. *VI.* 8, 17

Now since the sum of the sides is given, and the vertical angle is given,

$\therefore$ the right-angled $\triangle\ ASP$ can be constructed;

$\therefore$ PA and PS are known;

$\therefore$ PD is known.

Again, since $\angle$s ADH, AEH are right,

$\therefore$ the four points D, A, E, H are concyclic; *III.* 22

$\therefore$ $PA \cdot PD = PE \cdot PH$; *III.* 35, *Cor.*

$\therefore$ EH is divided externally at P, so that the rectangle contained by its segments $=$ a given rectangle.

Now $EH = AX$, the given perpendicular;

$\therefore$ PH and PE are known; *VI.* 29

$\therefore$ the point H can be determined.

The base BC of the required triangle is therefore that part, intercepted by the sides of the given vertical angle, of the tangent drawn from H to the circle whose centre is A and radius AX.

The synthesis may, with a little difficulty, be obtained by reversing the preceding analysis.

(2) When the difference of the sides is given.

About $\triangle\ ABC$ circumscribe a circle; draw, as before, the diameter PQ bisecting the base BC at H, and join AQ. From Q draw $QU \perp AB$, $AE \perp PQ$, and $AX \perp BC$; produce

BA to C', making $AC' = AC$; join CC', and let UH meet AQ at F.

Then $AU = \frac{1}{2}(AB - AC)$, $BU = \frac{1}{2}(AB + AC)$, and AQ bisects the angle adjacent to $\angle BAC$, by the twenty-fifth deduction of Book III.

But $BC' = AB + AC$;

$\therefore$ U is the middle point of BC';

$\therefore$ UH is $\parallel$ $C'C$, since H bisects BC; *App. I.* 1

$\therefore$ UF is $\perp$ AQ, since $C'C$ is $\perp$ AQ;

$\therefore$ $QA \cdot QF = QU^2$. *VI.* 8, 17

Now since the difference of the sides is given, and the vertical angle is given,

$\therefore$ the right-angled $\triangle$ AUQ can be constructed;

$\therefore$ QA and QU are known;

$\therefore$ QF is known.

Again since $\angle$s AFH, AEH are right,

$\therefore$ the four points F, A, E, H are concyclic; *III.* 22

$\therefore$ $QA \cdot QF = QE \cdot QH$; *III.* 35, *Cor.*

$\therefore$ EH is divided externally at Q, so that the rectangle contained by its segments = a given rectangle.

Now $EH = AX$, the given perpendicular;

$\therefore$ QH and QE are known; *VI.* 29

$\therefore$ the point H can be determined.

The base BC of the required triangle is obtained as before.

46. Let BC be the given base.

Divide BC internally at D and externally at D' in the ratio of the sides; on DD' as diameter describe a circle.

It will be shown in the seventh example of Loci, Book VI., that any point, the ratio of whose distances from B and C is equal to the given ratio, lies on the $\bigcirc^{ce}$ of this circle. Hence the vertex of the required triangle will be on this $\bigcirc^{ce}$. The vertex therefore will be determined by drawing a straight line $\parallel$ BC, and at a distance from it = the given perpendicular.

47. Since the rectangle contained by the sides = the rectangle contained by the perpendicular and the diameter of the circumscribed circle, the diameter of the circumscribed circle may therefore be found, and the circle described. In it a chord may be placed = the given base, and the vertex of the required triangle determined by drawing a straight line

$\parallel$ the given base, and at a distance from it = the given perpendicular.

48. Place the segments contiguous to each other and in the same straight line, and the base of the triangle is found.
Describe a circle as in the forty-sixth deduction, and from the point of section of the base draw a perpendicular to the base, which will cut the circle at the vertex of the required triangle.

49. The following is the solution usually given of this problem in French text-books. See Lafremoire's *Théorèmes et Problèmes de Géométrie Élémentaire* (1844), p. 84, or Catalan's subsequent editions of it.
Let a, b, c denote the sides of the required triangle, a', b', c' the corresponding given perpendiculars.

$$\text{The area of the triangle} = \tfrac{1}{2}\,aa' = \tfrac{1}{2}\,bb' = \tfrac{1}{2}\,cc';$$

$$\therefore \quad aa' = bb' = cc';$$

$$\therefore \quad \frac{a}{\frac{1}{a'}} = \frac{b}{\frac{1}{b'}} = \frac{c}{\frac{1}{c'}};$$

$$\therefore \quad \frac{a}{b'} = \frac{b}{a'} = \frac{c}{\frac{a'b'}{c'}}.$$

Hence a, b, c are proportional to b', a', $\dfrac{a'b'}{c'}$,
that is, to b', a', and a fourth proportional to c', a', b'.
Construct a triangle DEF whose sides are b', a', and this fourth proportional. The required triangle ABC will be similar to DEF, and its sides will be to those of DEF in the ratio of the perpendiculars of the two triangles.

50. Let AI be the distance of the vertex from the centre of the inscribed circle.

Produce AI to D, so that AD may be a fourth proportional to AI and the sides containing the vertical angle. On ID as diameter describe a circle; between A and the $\bigcirc^{ce}$ place straight lines AB, AC equal to the sides containing the vertical angle; and join BC.

ABC is the required triangle.

Let AC meet the $\bigcirc^{ce}$ of the circle again at E; join the centre O to E and B; join also BD, CI.

Then $AE \cdot AC = AI \cdot ID$, *III.* 35, *Cor.*

$= AB \cdot AC;$ *VI.* 16

$\therefore \quad AE = AB.$

Hence $\triangle$s ABC, AEC are congruent, *I.* 8

and AD bisects $\angle BAC$.

Now since $AI : AC = AB : AD$, and $\angle IAC = \angle BAD$,

$\therefore$ $\triangle$s AIC, ABD are similar, *VI.* 6

and $\angle ACI = \angle ADB$,

$= \angle BCI;$ *III.* 21

$\therefore$ CI bisects $\angle ACB$;

$\therefore$ I is the centre of the circle inscribed in $\triangle ABC$.

[This solution is taken from Leslie's *Geometrical Analysis and Geometry of Curve Lines* (1821), p. 22.]

TRANSVERSALS.

1. Let H, K, L be the middle points of the sides BC, CA, AB of the triangle.

   Then $AL \cdot BH \cdot CK = LB \cdot HC \cdot KA$;

   $\therefore$ AH, BK, CL are concurrent. *App. VI.* 2

2. See the notation adopted on p. 352 of *Euclid*.

   $AB : BC = AP : CP$, and $AB \cdot CP = BC \cdot AP$; *VI.* 3, 16

   $BC : CA = BQ : AQ$, and $BC \cdot AQ = CA \cdot BQ$; *VI.* 3, 16

   $CA : AB = CN : BN$, and $CA \cdot BN = AB \cdot CN$. *VI.* 3, 16

   By multiplication, and cancelling common factors,

   $AQ \cdot BN \cdot CP = QB \cdot NC \cdot PA$;

   $\therefore$ AN, BP, CQ are concurrent. *App. VI.* 2

3. $AB : BC = AP' : CP'$, and $AB \cdot CP' = BC \cdot AP'$; *VI.* A, 16

   $BC : CA = BQ' : AQ'$, and $BC \cdot AQ' = CA \cdot BQ'$; *VI.* A, 16

   $CA : AB = CN : BN$, and $CA \cdot BN = AB \cdot CN$. *VI.* 3, 16

   By multiplication, and cancelling common factors,

   $AQ' \cdot BN \cdot CP' = Q'B \cdot NC \cdot P'A$;

   $\therefore$ AN, BP', CQ' are concurrent. *App. VI.* 2

4. Because the points B, Z, Y, C are concyclic,

   $\therefore \quad AB \cdot AZ = AC \cdot AY.$ *III.* 35, *Cor.*

   Similarly $BC \cdot BX = BA \cdot BZ$,

   and $CA \cdot CY = CB \cdot CX$.

   By multiplication, and cancelling common factors,

   $AZ \cdot BX \cdot CY = ZB \cdot XC \cdot YA$;

   $\therefore$ AX, BY, CZ are concurrent. *App. VI.* 2

5. Let AL, BK, CF meet BC, CA, AB at R, S, T.
Then $BR : RC = AB^2 : AC^2$, by the third deduction from VI. 8, and the first deduction from VI. 20;
$\therefore\ BR \cdot AC^2 = RC \cdot AB^2$.
Because $\triangle$s CSK, ASB are similar,
$\therefore\ CS : AS = CK : AB = AC : AB$;
$\therefore\ CS \cdot AB = AS \cdot AC$.
Because $\triangle$s ATC, BTF are similar,
$\therefore\ AT : BT = AC : BF = AC : AB$;
$\therefore\ AT \cdot AB = BT \cdot AC$.
By multiplication, and cancelling common factors,
$AT \cdot BR \cdot CS = TB \cdot RC \cdot SA$;
$\therefore\ AL$, BK, CF are concurrent. *App. VI.* 2

6. Let BE, CD (fig. to VI. 2) intersect at O.
Join AO, and let it meet BC at H.
Then $AD : DB = AE : EC$, and $AD \cdot EC = DB \cdot AE$. *VI.* 2, 16
But $AD \cdot BH \cdot CE = DB \cdot HC \cdot EA$; *App. VI.* 2
$\therefore$ by division, $BH = HC$.
Conversely: Let AH be the median from A;
through any point O on AH let BOE, COD be drawn to meet AC, AB at E, D.
Then $AD \cdot BH \cdot CE = DB \cdot HC \cdot EA$, *App. VI.* 2
But $BH = HC$;
$\therefore$ by division, $AD \cdot CE = DB \cdot EA$;
$\therefore$ $AD : DB = AE : EC$. *VI.* 16

7. To prove N', P, Q collinear.
$AB : BC = AP : CP$, and $AB \cdot CP = BC \cdot AP$; *VI.* 3, 16
$BC : CA = BQ : AQ$, and $BC \cdot AQ = CA \cdot BQ$; *VI.* 3, 16
$CA : AB = CN' : BN'$, and $CA \cdot BN' = AB \cdot CN'$. *VI.* A, 16
By multiplication, and cancelling common factors,
$AQ \cdot BN' \cdot CP = QB \cdot N'C \cdot PA$;
$\therefore\ N'$, P, Q are collinear. *App. VI.* 1

8. To prove N', P', Q' collinear.
$AB : BC = AP' : CP'$, and $AB \cdot CP' = BC \cdot AP'$; *VI.* A, 16
$BC : CA = BQ' : AQ'$, and $BC \cdot AQ' = CA \cdot BQ'$; *VI.* A, 16
$CA : AB = CN' : BN'$, and $CA \cdot BN' = AB \cdot CN'$. *VI.* A. 16
By multiplication, and cancelling common factors,
$AQ' \cdot BN' \cdot CP' = Q'B \cdot N'C \cdot P'A$;
$\therefore\ N'$, P', Q' are collinear. *App. VI.* 1

9. Let ABC be the triangle, and let the tangents to the circle at A, B, C meet the opposite sides respectively at D, E, F.

Because $\triangle$s ABD, CAD are mutually equiangular, *III.* 32

$\therefore\ BD : AD = AB : CA$,

and $AD : CD = AB : CA$. *VI.* 4

Compound these two proportions;

then $BD : CD = AB^2 : CA^2$;

$\therefore\ BD \cdot CA^2 = CD \cdot AB^2$.

Similarly $CE \cdot AB^2 = AE \cdot BC^2$,

and $AF \cdot BC^2 = BF \cdot AC^2$.

By multiplication, and cancelling common factors,

$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$;

$\therefore\ D, E, F$ are collinear. *App. VI.* 1

10. Let AN be the bisector of $\angle A$, and let the perpendicular to AN at its middle point meet BC at D.

Join AD, and about $\triangle ABC$ circumscribe a circle.

Then $DA = DN$; $\therefore \angle DAN = \angle DNA$.

But $\angle DAN = \angle DAC + \angle CAN$,

and $\angle DNA = \angle B + \angle BAN$;

$\therefore\ \angle DAC + \angle CAN = \angle B + \angle BAN$;

$\therefore\ \angle DAC = \angle B$;

$\therefore\ AD$ is a tangent to the circumscribed circle,

that is, the perpendicular to AN at its middle point meets BC where a tangent at A meets BC.

Hence also the perpendiculars to BP, CQ at their middle points meet CA, AB, where tangents at B, C meet CA, AB;

$\therefore$ these three points are collinear, by the preceding deduction.

11. For $O'B'' : OB'' = O'A' : OA$ (from $\triangle$s $O'A'B''$, OAB''); *VI.* 4

$\therefore\ O'B'' \cdot OA = OB'' \cdot O'A'$. *VI.* 16

Similarly $OB' \cdot O''A'' = O''B' \cdot OA$,

and $O''B \cdot O'A' = O'B \cdot O''A''$.

By multiplication, and cancelling common factors,

$OB' \cdot O'B'' \cdot O''B = O''B' \cdot OB'' \cdot O'B$;

$\therefore\ B, B', B''$ are collinear. *App. VI.* 1

12. Let $ABCDE$ be any polygon, and let its sides AB, BC, CD, DE, EA be cut by a transversal in the points F, G, H, K, L respectively: to prove

$AF \cdot BG \cdot CH \cdot DK \cdot EL = BF \cdot CG \cdot DH \cdot EK \cdot AL$.

Join A, one of the vertices of the polygon, with C, D, the

other vertices with which it is not joined already, and let AC, AD meet the transversal in M, N.

From $\triangle\ ABC$ cut by the given transversal there results

$AF \cdot BG \cdot CM = BF \cdot CG \cdot AM$; *App. VI.* 1

from $\triangle\ ACD$ cut by the given transversal there results

$AM \cdot CH \cdot DN = CM \cdot DH \cdot AN$; *App. VI.* 1

from $\triangle\ ADE$ cut by the given transversal there results

$AN \cdot DK \cdot EL = DN \cdot EK \cdot AL$. *App. VI.* 1

By multiplication, and cancelling common factors,

$AF \cdot BG \cdot CH \cdot DK \cdot EL = BF \cdot CG \cdot DH \cdot EK \cdot AL$.

13. Let $ABCDEF$ be a hexagon inscribed in a circle, and let the opposite sides AB, DE meet in G; BC, EF in H; CD, FA in K: to prove G, H, K collinear.

Produce the alternate sides EF, AB to meet at L; AB, CD to meet at M; and CD, EF at N.

From $\triangle\ LMN$ cut by the transversals BC, DE, FA, the three other alternate sides, there result

$LB \cdot MC \cdot NH = MB \cdot NC \cdot LH$,

$LG \cdot MD \cdot NE = MG \cdot ND \cdot LE$,

$LA \cdot MK \cdot NF = MA \cdot NK \cdot LF$. *App. VI.* 1

By multiplication, and remembering that

$LA \cdot LB = LE \cdot LF$, $MC \cdot MD = MA \cdot MB$, $NE \cdot NF = NC \cdot ND$, *III.* 35, *Cor.*

there results $LG \cdot MK \cdot NH = MG \cdot NK \cdot LH$;

$\therefore$ G, H, K are collinear. *App. VI.* 1

14. From $\triangle\ BCF$ cut by the transversal AD, there results

$BD \cdot CO \cdot FA = CD \cdot FO \cdot BA$;

from $\triangle\ CAD$ cut by the transversal BE, there results

$CE \cdot AO \cdot DB = AE \cdot DO \cdot CB$;

from $\triangle\ ABE$ cut by the transversal CF, there results

$AF \cdot BO \cdot EC = BF \cdot EO \cdot AC$.

By multiplication, and remembering that

$FA \cdot DB \cdot EC = FB \cdot DC \cdot EA$,

$$AO \cdot BO \cdot CO \cdot AF \cdot BD \cdot CE = DO \cdot EO \cdot FO \cdot AB \cdot BC \cdot CA;$$

$$\therefore AO \cdot BO \cdot CO : DO \cdot EO \cdot FO = AB \cdot BC \cdot CA : AF \cdot BD \cdot CE.$$

15. Draw $AX \perp BC$.

Then $AC^2 = AD^2 + CD^2 - 2\,CD \cdot DX$; *II.* 13

$\therefore AC^2 \cdot BD = AD^2 \cdot BD + CD^2 \cdot BD - 2\,BD \cdot DC \cdot DX$ (1).

Again, $AB^2 = AD^2 + BD^2 \pm 2\,BD \cdot DX$. *II.* 12, 13

$\therefore AB^2 \cdot CD = AD^2 \cdot CD + BD^2 \cdot CD \pm 2\,BD \cdot DC \cdot DX$ (2).

From (1) and (2) by addition or subtraction,

$$AC^2 \cdot BD \pm AB^2 \cdot CD$$
$$= AD^2 \cdot (BD \pm CD) + BD \cdot CD\,(CD \pm BD),$$
$$= AD^2 \cdot BC \pm BD \cdot DC \cdot BC.$$

Appendix, II. 1.

If D be the middle point of BC,

then $AC^2 \cdot BD + AB^2 \cdot BD = 2\,AD^2 \cdot BD + 2\,BD^3$.

Divide both sides by BD;

then $AC^2 + AB^2 = 2\,AD^2 + 2\,BD^2$.

First deduction of Book II.

If $AB = AC$, and D lie on BC,

then $AB^2 \cdot BD + AB^2 \cdot CD = AD^2 \cdot BC + BD \cdot DC \cdot BC$;

$\therefore \quad AB^2 \cdot (BD + DC) \quad = (AD^2 + BD \cdot DC) \cdot BC$.

Divide both sides by BC, which $= BD + DC$;

then $AB^2 = AD^2 + BD \cdot DC$;

$\therefore \quad AB^2 - AD^2 = BD \cdot DC$.

If $AB = AC$, and D lie on BC produced,

then $AB^2 \cdot BD - AB^2 \cdot CD = AD^2 \cdot BC - BD \cdot DC \cdot BC$;

$\therefore \quad AB^2\,(BD - CD) = (AD^2 - BD \cdot DC) \cdot BC$.

Divide both sides by BC, which $= BD - CD$;

$\therefore \qquad AB^2 = AD^2 - BD \cdot DC$;

$\therefore AD^2 - AB^2 = BD \cdot DC$.

VI. B.

If AD bisect the interior vertical angle at A,

then $AB : AC = BD : CD$, and $AC \cdot BD = AB \cdot CD$. *VI.* 3, 16

Hence $AC \cdot AB \cdot CD + AB \cdot AC \cdot BD$

$$= (AD^2 + BD \cdot DC) \cdot BC;$$

$\therefore AB \cdot AC \cdot (CD + BD) = (AD^2 + BD \cdot DC) \cdot BC$.

Divide both sides by BC, which $= CD + BD$;

$\therefore AB \cdot AC = AD^2 + BD \cdot DC$;

$\therefore \quad AD^2 \;= AB \cdot AC - BD \cdot DC$.

If AD bisect the exterior vertical angle at A,

then $AB : AC = BD : CD$, and $AC \cdot BD = AB \cdot CD$. *VI.* A, 16

Hence $AC \cdot AB \cdot CD - AB \cdot AC \cdot BD$

$$= (AD^2 - BD \cdot DC) \cdot BC;$$

$\therefore \quad AB \cdot AC \cdot (BD - CD) = (BD \cdot DC - AD^2) \cdot BC$.

Divide both sides by BC, which $= BD - CD$;

$\therefore\ AB \cdot AC = BD \cdot DC - AD^2$;

$\therefore\ AD^2 = BD \cdot DC - AB \cdot AC$.

APPENDIX, VI. 8.

If the two bisectors in the preceding deduction be denoted by AD_1 and AD_2,

then $AB \cdot AC - BD_1 \cdot D_1C = AD_1^2$,

and $BD_2 \cdot D_2C - AB \cdot AC = AD_2^2$;

$\therefore\ BD_2 \cdot D_2C - BD_1 \cdot D_1C = AD_1^2 + AD_2^2$,

$= D_1D_2^2$. *I.* 47

16. For $AF \cdot AF' = AE \cdot AE'$,

$BD \cdot BD' = BF \cdot BF'$,

$CE \cdot CE' = CD \cdot CD'$; *III.* 35, or *Cor.*

$\therefore$ by multiplication we obtain the relation specified.

17. Since $\triangle$s AII_2, AI_3I_1 are mutually equiangular, by (13) of the nineteenth deduction of Book IV.;

$\therefore\ AI : AI_2 = AI_3 : AI_1$; *VI.* 4

$\therefore\ AI \cdot AI_1 = AI_2 \cdot AI_3$. *VI.* 16

Again, $\triangle$s ABI_3, AI_2C are mutually equiangular, by (14) (a) of the nineteenth deduction of Book IV.;

$\therefore\ AB : AI_3 = AI_2 : AC$; *VI.* 4

$\therefore\ AI_2 \cdot AI_3 = AB \cdot AC$; *VI.* 16

$\therefore\ AI \cdot AI_1 = AB \cdot AC$.

Similarly $BI \cdot BI_2 = BA \cdot BC$,

and $CI \cdot CI_3 = CA \cdot CB$;

$\therefore\ AI \cdot BI \cdot CI \cdot AI_1 \cdot BI_2 \cdot CI_3 = (AB \cdot BC \cdot CA)^2$;

$\therefore\ AI \cdot BI \cdot CI : AB \cdot BC \cdot CA = AB \cdot BC \cdot CA : AI_1 \cdot BI_2 \cdot CI_3$.

18. (1) For $\dfrac{AF}{BF} \cdot \dfrac{BD}{CD} \cdot \dfrac{CE}{AE} = \dfrac{s-a}{s-b} \cdot \dfrac{s-b}{s-c} \cdot \dfrac{s-c}{s-a} = 1$.

(2) For $\dfrac{AF_1}{BF_1} \cdot \dfrac{BD_1}{CD_1} \cdot \dfrac{CE_1}{AE_1} = \dfrac{s}{s-c} \cdot \dfrac{s-c}{s-b} \cdot \dfrac{s-b}{s} = 1$.

Hence also (3) and (4).

(5) For $\dfrac{AF_3}{BF_3} \cdot \dfrac{BD_1}{CD_1} \cdot \dfrac{CE_2}{AE_2} = \dfrac{s-b}{s-a} \cdot \dfrac{s-c}{s-b} \cdot \dfrac{s-a}{s-c} = 1$.

(6) For $\dfrac{AF_2}{BF_2} \cdot \dfrac{BD}{CD} \cdot \dfrac{CE_3}{AE_3} = \dfrac{s-c}{s} \cdot \dfrac{s-b}{s-c} \cdot \dfrac{s}{s-b} = 1$.

Hence also (7) and (8).

(9) Let Q_1 be the point where DE meets AB.

Then $\dfrac{AQ_1}{BQ_1} = \dfrac{CD}{BD} \cdot \dfrac{AE}{CE} = \dfrac{(s-c)(s-a)}{(s-b)(s-c)} = \dfrac{s-a}{s-b}$.

Let Q_1' be the point where D_2E_1 meets AB.

Then $\dfrac{AQ_1'}{BQ_1'} = \dfrac{CD_2}{BD_2} \cdot \dfrac{AE_1}{CE_1} = \dfrac{(s-a)\,s}{s\,(s-b)} = \dfrac{s-a}{s-b}$;

$\therefore$ Q_1 and Q_1' are the same point.

Hence also (10) and (11).

(12) Let Q_2 be the point where D_1E_2 meets AB.

Then $\dfrac{AQ_2}{BQ_2} = \dfrac{CD_1}{BD_1} \cdot \dfrac{AE_2}{CE_2} = \dfrac{(s-b)\,(s-c)}{(s-c)\,(s-a)} = \dfrac{s-b}{s-a}$.

Let Q_2' be the point where D_3E_3 meets AB.

Then $\dfrac{AQ_2'}{BQ_2'} = \dfrac{CD_3}{BD_3} \cdot \dfrac{AE_3}{CE_3} = \dfrac{s\,(s-b)}{(s-a)\,s} = \dfrac{s-b}{s-a}$;

$\therefore$ Q_2 and Q_2' are the same point.

Hence also (13) and (14).

(15) Let Q_3 be the point where NP meets AB.

Then $\dfrac{AQ_3}{BQ_3} = \dfrac{CN}{BN} \cdot \dfrac{AP}{CP}$,

$= \dfrac{ab}{b+c} \cdot \dfrac{bc}{a+c} \cdot \dfrac{b+c}{ac} \cdot \dfrac{a+c}{ab}$,

by the ninth deduction from VI. 3,

$= \dfrac{b}{a}$.

But I_1I_2, since it bisects the exterior angle at C, meets AB at a point the ratio of whose distances from A and $B = \dfrac{b}{a}$; *VI.* A

$\therefore$ I_1I_2 passes through Q_3.

Hence also (16) and (17).

19. Let $A'B'C'$ be the triangle formed by drawing through A, B, C parallels to the opposite sides, $A'B'$ being $\parallel AB$, $B'C' \parallel BC$, $C'A' \parallel CA$.

From $\triangle\ BCE_2$ cut by the transversal $AI'D_1$ there results

$BI' : I'E_2 = BD_1 \cdot CA : CD_1 \cdot AE_2$, *App. VI.* 1, *Cor.* 1

$= (s-c)\,b \ : (s-b)\,(s-c)$,

$= b : s-b$.

If through I' there be drawn a parallel to BC, meeting AC in U and AX, which is $\perp BC$, in V, this parallel will divide CE_2, or $s-a$, into two segments proportional to b and $s-b$. Hence $CU = \dfrac{b\,(s-a)}{s}$, and $AU = b - \dfrac{b\,(s-a)}{s} = \dfrac{ab}{s}$.

Now $AC : AX = AU : AV$;

$\therefore \quad b : \dfrac{2\,rs}{a} = \dfrac{ab}{s} : AV$;

$\therefore \quad AV = 2\,r$,

that is, the distance of I' from $B'C'$ is $2r$.

Hence also the distance of I' from $C'A'$ and $A'B'$ is $2r$;

$\therefore$ I' is the centre of the circle inscribed in $\triangle\, A'B'C'$.

Similarly it may be proved that

the distance of I_1' from $B'C'$, $C'A'$, $A'B'$ is $2r_1$,

the distance of I_2' from $B'C'$, $C'A'$, $A'B'$ is $2r_2$,

the distance of I_3' from $B'C'$, $C'A'$, $A'B'$ is $2r_3$.

20. Q_1, N_1, P_1 are collinear.

For $\dfrac{AQ_1}{BQ_1} \cdot \dfrac{BN_1}{CN_1} \cdot \dfrac{CP_1}{AP_1} = \dfrac{s-a}{s-b} \cdot \dfrac{s-b}{s-c} \cdot \dfrac{s-c}{s-a} = 1.$

Q_2, N_2, P_2 are collinear.

For $\dfrac{AQ_2}{BQ_2} \cdot \dfrac{BN_2}{CN_2} \cdot \dfrac{CP_2}{AP_2} = \dfrac{s-b}{s-a} \cdot \dfrac{s-c}{s-b} \cdot \dfrac{s-a}{s-c} = 1.$

Q_3, N_3, P_3 are collinear.

For $\dfrac{AQ_3}{BQ_3} \cdot \dfrac{BN_3}{CN_3} \cdot \dfrac{CP_3}{AP_3} = \dfrac{b}{a} \cdot \dfrac{c}{b} \cdot \dfrac{a}{c} = 1.$

Again consider $\triangle\, AQ_1P_1$ cut by the transversals p and q.

Let T be the point where p meets Q_1P_1 or n.

$$\text{Then } \frac{Q_1T}{P_1T} = \frac{Q_1Q_2 \cdot AP_2}{AQ_2 \cdot P_1P_2} = \frac{c^2}{b-a} \cdot \frac{b(s-c)}{c-a} \cdot \frac{b-a}{c(s-b)} \cdot \frac{c-a}{b^2},$$

$$= \frac{c(s-c)}{b(s-b)}.$$

Let T' be the point where q meets Q_1P_1 or n.

$$\text{Then } \frac{Q_1T'}{P_1T'} = \frac{Q_1Q_3 \cdot AP_3}{AQ_3 \cdot P_1P_3} = \frac{c(s-c)}{b-a} \cdot \frac{bc}{c-a} \cdot \frac{b-a}{bc} \cdot \frac{c-a}{b(s-b)},$$

$$= \frac{c(s-c)}{b(s-c)};$$

$\therefore$ T and T' are the same point.

$\therefore$ n, p, q are concurrent.

HARMONICAL PROGRESSION.

1. Let AB (fig. to VI. 30) be cut in extreme and mean ratio at C.

Then $\quad AB : AC = AC : BC$;

$\therefore\; AB - AC : AB = AC - BC : AC$;

$$\therefore \quad AC - BC = \frac{AC\cdot(AB - AC)}{AB} = \frac{AC \cdot BC}{AC + BC}.$$

2. Let AB and AD (fig. to VI. 10) be the mean and the greater extreme.

Draw any line AE making an angle with AD; take any point F in it, and join DF. Through B draw $BH \parallel DF$, and make $FG = FH$; join GB, and draw $FC \parallel GB$, meeting AB at C.

Because $AC : CB = AF : FG$, *VI.* 2
$= AF : FH$,
$= AD : DB$; *VI.* 2

$\therefore$ AD, AB, AC are in harmonical progression.

3. Let AB and AC (fig. to VI. 10) be the mean and the less extreme.

Draw any line AE making an angle with AB; take any point F in it, and join CF. Through B draw $BG \parallel CF$, and make $FH = FG$; join HB, and draw $FD \parallel HB$, meeting AB produced at D.

Because $AC : CB = AF : FG$, *VI.* 2
$= AF : FH$,
$= AD : DB$; *VI.* 2

$\therefore$ AD, AB, AC are in harmonical progression.

4. Let AC and AD (fig. to VI. 10) be the two extremes.

Draw any line AE making an angle with AD; take any point F in it, and join CF, DF. In FE take any point K, and from FA cut off $FL = FK$; through K and L draw KM, LM respectively parallel to CF, DF. If M is not situated on CD, join FM, and let FM, produced if necessary, cut CD at B. AB is the required mean.

Through B draw BG, BH respectively parallel to KM, LM.

Then $FG : FK = FB : FO$, *VI.* 4
$= FH : FL$. *VI.* 4

But $FK = FL$; $\therefore$ $FG = FH$.

Now $AC : CB = AF : FG$, *VI.* 2
$= AF : FH$,
$= AD : DB$; *VI.* 2

$\therefore$ AD, AB, AC are in harmonical progression.

5. Let EF be a chord, AB a diameter perpendicular to it; from G, any point in the $\bigcirc^{ce}$, let EG and FG be drawn; let FG cut

AB in C, and EG produced cut AB produced in D: to prove that A, C, B, D form a harmonic range.

Join AG, BG.

Since AB is $\perp$ EF, $\therefore$ arc AE = arc AF;

$\therefore$ $\angle AGE = \angle AGF$, or $\angle CGE$ is bisected by AG.

Hence BG, which is $\perp$ AG, bisects $\angle CGD$;

$\therefore$ $DB : BC = DG : GC$, *VI.* 3

and $DG : GC = DA : AC$; *VI.* A

$\therefore$ $DB : BC = DA : AC$;

$\therefore$ DC is cut harmonically at A and B;

$\therefore$ AB is cut harmonically at C and D. *App. VI.* 4

6. Let FCG be a circle, EF, EG two tangents to it from E, and FG the chord of contact. Let AB, a third tangent to the circle at C, cut EF at A, EG at B, and FG produced at D: to prove that A, C, B, D form a harmonic range.

From A draw $AH \parallel EG$, meeting FG at H.

Since $\triangle$ EFG is isosceles, *III.* 17, *Cor.*

$\therefore$ $\triangle$ AFH is isosceles, and $AH = AF = AC$; *III.* 17, *Cor.*

and $BG = BC$. *III.* 17, *Cor.*

Now since $\triangle$s AHD, BGD are similar,

$\therefore$ $AD : DB = AH : BG$,

$= AC : CB$;

$\therefore$ A, C, B, D form a harmonic range.

7. Let AF meet DE at G, and BC at H.

Then $HA : AG = BA : AD$ (from $\triangle$s BHA, DGA), *VI.* 4

$= BC : DE$ (from $\triangle$s ABC, ADE), *VI.* 4

$= BF : FE$ (from $\triangle$s BFC, EFD), *VI.* 4

$= HF : FG$ (from $\triangle$s BHF, EGF); *VI.* 4

$\therefore$ A, G, F, H form a harmonic range.

Again, $CH : EG = HA : AG$ (from $\triangle$s AHC, AGE), *VI.* 4

$= HF : FG$,

$= BH : EG$ (from $\triangle$s BHF, EGF); *VI.* 4

$\therefore$ $BH = CH$.

8. Let DE, DF be two tangents to a circle from a point D, and EF the chord of contact. From D let the secant DBA be drawn cutting the $\bigcirc^{ce}$ at B and A, and EF at C: to prove that A, C, B, D form a harmonic range.

Since $\triangle$ DEF is isosceles,

$\therefore$ $ED^2 = CD^2 + EC \cdot CF$, by the first deduction of Book II.

$\therefore$ $AD \cdot DB = CD^2 + AC \cdot CB$; *III.* 35, and *Cor.*

$\therefore AD \cdot DB - AC \cdot CB = CD^2;$

$\therefore A, C, B, D$ form a harmonic range. *App. VI.* 8

9. Let EC meet the $\bigcirc^{ce}$ at F and G. Bisect AB at O; draw $OH \perp FG$, and therefore bisecting FG. *III.* 3

Since the angles at H and D are right,

$\therefore$ the points O, H, D, E are concyclic;

$\therefore HC \cdot CE = OC \cdot CD,$ *III.* 35

$= AC \cdot CB,$ *App. VI.* 7

$= GC \cdot CF;$ *III.* 35

$\therefore G, C, F, E$ form a harmonic range. *App. VI.* 7

10. For $DE = \text{radius} = \frac{1}{2}(AD + DB);$

$DF^2 = AD \cdot DB;$ *III,* 3, 35

and $DC = \dfrac{AD \cdot DB}{DE} = \dfrac{2\,AD \cdot DB}{2\,DE} = \dfrac{2\,AD \cdot DB}{AD + DB}.$

Again, $DF^2 = AD \cdot DB = ED \cdot DC,$ *III.* 35

which proves App. VI. 6, Cor. 2.

11. For $DE = \frac{1}{2}(AD + DB);$

$DF^2 = AD \cdot DB.$ *III.* 36

Now since $\triangle\, DEF$ is right-angled, *III.* 18

$\therefore EC \cdot ED = EF^2,$ *VI.* 8, *Cor.*

$= EB^2;$

and $EA = EB;$

$\therefore A, C, B, D$ form a harmonic range; *App. VI.* 5

$\therefore DC$ is the harmonic mean between DA and DB.

Again, $DF^2 = ED \cdot DC,$ *VI.* 8, *Cor.*

which proves App. VI. 6, Cor. 2.

12. Let $O \cdot ACBD$ be a pencil, FH a transversal $\parallel OA$, and bisected at G by OB, F being on OC, and H on OD: to prove that the pencil cuts any transversal $ACBD$ harmonically.

Through B draw $KL \parallel OA$, meeting OC, OD, produced if necessary, at K, L respectively.

Then $OB : BK = OG : GF,$ *VI.* 4

$= OG : GH,$

$= OB : BL;$ *VI.* 4

$\therefore BK = BL;$

$\therefore AO : BK = AO : BL.$

Now $AO : BK = AC : BC,$ *VI.* 4

and $AO : BL = AD : BD;$ *VI.* 4

$\therefore AC : BC = AD : BD;$

$\therefore A, C, B, D$ form a harmonic range,

and the pencil $O \cdot ACBD$ is harmonic.

13. Let $O \cdot ACBD$ (fig. to the preceding deduction) be a harmonic pencil, and A, C, B, D a harmonic range; let FH, which is $\parallel OA$, intersect OB in G: to prove $GF = GH$.

Through B draw $KL \parallel OA$, meeting OC, OD, produced if necessary, at K, L respectively.

Since A, C, B, D form a harmonic range.

$\therefore \quad AC : BC = AD : BD.$

But $AC : BC = AO : BK$, *VI.* 4

and $AD : BD = AO : BL$; *VI.* 4

$\therefore \quad AO : BK = AO : BL$;

$\therefore \quad BK = BL.$

Hence $GF = GH.$

14. The bisector of the interior vertical angle cuts the base internally in the ratio of the sides, and the bisector of the exterior vertical angle cuts the base externally in the ratio of the sides.

15. Let A, C, B, D be a harmonic range, and $O \cdot ACBD$ a harmonic pencil, and let $\angle AOB$ be right.

Through B draw $KL \parallel OA$, meeting OC, OD, produced if necessary, at K, L respectively.

Then, as in the thirteenth deduction of this set, $BK = BL$.

Since BK is $\parallel OA$, and $\angle AOB$ is right,

$\therefore \angle OBK$ is right; *I.* 29

$\therefore \angle OBL$ is right;

$\therefore \triangle$s OBK, OBL are congruent, *I.* 4

and OB bisects $\angle COD$.

Hence OA bisects the angle supplementary to $\angle COD$.

16. Let the pencil $O \cdot ACBD$ divide the transversal $ACBD$ harmonically: to prove that it will divide the transversal $A'C'B'D'$ harmonically.

Through B' draw $K'L' \parallel OA$, meeting OC, OD, produced if necessary, at K', L' respectively.

Then $K'L'$ is bisected by OB, by the thirteenth deduction of this set;

$\therefore A'$, C', B', D' is a harmonic range, by the twelfth deduction of this set.

17. Let CC', BB' meet at O; join OA, OD.

If OD does not pass through D', let it cut $AC'B'$ at D''.

Since the pencil $O \cdot ACBD$ cuts the transversal $ACBD$ harmonically,

$\therefore$ it will cut the transversal $AC'B'D''$ harmonically;

$\therefore$ A, C', B', D'' form a harmonic range;

$\therefore$ $AC' : B'C' = AD'' : B'D''$.

But since A, C', B', D' form a harmonic range,

$\therefore$ $AC' : B'C' = AD' : B'D'$;

$\therefore$ $AD'' : B'D'' = AD' : B'D'$,

which is impossible unless D' and D'' be the same point, by the first deduction of Book VI.

Hence CC', BB', DD' are concurrent.

18. From $\triangle\, EAB$ cut by the transversal GFD there results

$AD : DB = FE \cdot GA : BF \cdot EG$; *App. VI.* 1, *Cor.* 1

from $\triangle\, EAB$ and the three concurrent straight lines EC, AF, BG, there results

$AC : CB = FE \cdot GA : BF \cdot EG$; *App. VI.* 2, *Cor.* 1

$\therefore$ $AC : CB = AD : DB$;

$\therefore$ A, C, B, D form a harmonic range.

19. If $\angle\, ACB - \angle\, ABC =$ a right angle,

then $\angle\, ADB - \angle\, ADC =$ a right angle.

But $\angle\, ADB - \angle\, AED =$ a right angle;

$\therefore$ $\angle\, ADC = \angle\, AED$;

$\therefore$ $AD = AE$, and $AD^2 = AE^2$;

$\therefore$ $BE \cdot EC - BA \cdot AC = BA \cdot AC - BD \cdot DC$;

$\therefore$ $BE \cdot EC$, $BA \cdot AC$, $BD \cdot DC$ are in arithmetical progression.

Again, since $BD : DC = BA : AC = BE : EC$, *VI.* 3, A

$\therefore$ the three rectangles $BD \cdot DC$, $BA \cdot AC$, $BE \cdot EC$ are similar figures.

Now if $\angle\, ACB$ is right, since $\triangle\, DAE$ is right-angled,

$\therefore$ $DC : CA = AC : CE$; *VI.* 8, *Cor.*

$\therefore$ $BD \cdot DC : BA \cdot AC = BA \cdot AC : BE \cdot EC$; *VI.* 22

$\therefore$ $BD \cdot DC$, $BA \cdot AC$, $BE \cdot EC$ are in geometrical progression.

Lastly, if $\angle\, BAC$ is right, then $\angle\, DAC =$ half a right angle;

$\therefore$ $\angle\, EAC =$ half a right angle;

$\therefore$ $CE : CD = AE : AD$; *VI.* 3

$\therefore$ $BE \cdot EC : BD \cdot DC = AE^2 : AD^2$. *VI.* 22

Now $BE \cdot EC - BA \cdot AC = AE^2$,

and $BA \cdot AC - BD \cdot DC = AD^2$;

$\therefore$ $BE \cdot EC$, $BA \cdot AC$, $BD \cdot DC$ are in harmonical progression.

[It may be noted that Dr Lardner's proof of the second part of this deduction is not quite sound.]

20. Let C be the centre of the circle, AB a side of the inscribed polygon K, CD a radius perpendicular to, and therefore bisecting AB in I.

If EF be drawn, touching the circle in D, and terminated by CA and CB produced, EF will be a side of the circumscribed polygon L of the same number of sides.

Also if AD be joined, and at the points A and B tangents be drawn to meet EF in G and H, AD and GH will be sides of the polygons M and N, of twice the number of sides.

Now of $\triangle$s CAI, CED, CAD, CGH the polygons K, L, M, N are the same equimultiples;

and $\triangle\, CAI : \triangle\, CAD = CI : CD$, *VI.* 1

$= CA : CE$, *VI.* 2

$= \triangle\, CAD : \triangle\, CED$;

$\therefore \quad K : M = M : L$.

Again, $\triangle\, CGH = 2\, \triangle\, CGD = CAGD$;

$\therefore \quad \triangle\, CED - \triangle\, CGH = \triangle\, AEG$,

and $\quad \triangle\, CGH - \triangle\, CAD = \triangle\, AGD$;

$\therefore \triangle\, CED - \triangle\, CGH : \triangle\, CGH - \triangle\, CAD$,

$= \triangle\, AEG : \triangle\, AGD$,

$= EG : GD$, *VI.* 1

$= EC : CD$, *VI.* 3

$= EC : CA$,

$= \triangle\, CED : \triangle\, CAD$; *VI.* 1

$\therefore$ $\triangle$s CED, CGH, CAD are in harmonical progression;

$\therefore$ L, N, M are in harmonical progression.

CENTRES OF SIMILITUDE.

1. When they touch externally, the point of contact is the internal centre of similitude. In that case the external centre is situated outside both circles.

   The truth of the last statement, and of some of those that follow, is seen by supposing the situation of either centre of similitude not to be as stated, and then showing that the supposed situation is inconsistent with Def. 5, on p. 352.

2. When they touch internally, the point of contact is the external centre of similitude. In that case the internal centre of similitude is inside both circles.

3. When the circles are exterior to each other, and have no common point, both centres of similitude are outside both circles.
   When the one circle is inside the other, and does not touch it, both centres of similitude are inside both circles.
4. When the circles intersect each other.
5. Let A and B be the centres of two circles which intersect at F and G, and let I, E be the internal and external centres of similitude.

   Join F to A, I, B, E.

   Then $AI : BI = AF : BF = AE : BE$;

   $\therefore$ FI bisects $\angle AFB$, and FE bisects the angle adjacent to $\angle AFB$. *VI.* 3, A
6. Let A, B be the centres of two circles, and let the direct common tangent MN intersect the line of centres AB produced in E.

   Join AM, BN.

   Then $\triangle$s AEM, BEN are mutually equiangular, since $\angle$s AME, BNE are right; *III.* 18

   $\therefore$ $AE : BE = AM : BN$; *VI.* 4

   $\therefore$ E is the external centre of similitude.

   Again, let the transverse common tangent PQ intersect the line of centres AB in I.

   Join AP, BQ.

   Then $\triangle$s AIP, BIQ are mutually equiangular, since $\angle$s API, BQI are right; *III.* 18

   $\therefore$ $AI : BI = AP : BQ$; *VI.* 4

   $\therefore$ I is the internal centre of similitude.
7. Let A and B be the centres of two circles whose radii are a and b, and let E be the external centre of similitude.
   From E draw EM tangent to the circle A.

   Join AM, and draw $BN \perp EM$.

   Since $\angle$s M and N are right, and $\angle E$ is common,

   $\therefore$ $\triangle$s AEM, BEN are mutually equiangular;

   $\therefore$ $AE : BE = AN : BN$, *VI.* 4

   $= a : BN$.

   But $AE : BE = a : b$;

   $\therefore$ $a : b = a : BN$;

   $\therefore$ $BN = b$, that is, N lies on the $\bigcirc^{ce}$ of circle B.

   Now since $\angle BNE$ is right, EN is tangent to the circle B.

   The proof for the internal centre of similitude is similar to the preceding.

8. See fig. on p. 251 of *Euclid.*

A is the external centre of similitude of the circles DEF, $D_1E_1F_1$; and A is the internal centre of similitude of the circles $D_2E_2F_2$, $D_3E_3F_3$, by the sixth deduction of this set. Similarly for B and C.

9. See fig. on p. 251 of *Euclid.*

The point N is the point of intersection of BC and II_1.
Now II_1 is the line of centres of the circles DEF, $D_1E_1F_1$, and BC is a transverse common tangent to them;
$\therefore$ by the sixth deduction of this set,
N is the internal centre of similitude of the circles DEF, $D_1E_1F_1$.
Similarly for P and Q.

10. See fig. on p. 251 of *Euclid.*

The point N' is the point of intersection of BC and I_2I_3.
Now I_2I_3 is the line of centres of the circles $D_2E_2F_2$, $D_3E_3F_3$, and BC is a direct common tangent to them;
$\therefore$ by the sixth deduction of this set,
N' is the external centre of similitude of the circles $D_2E_2F_2$, $D_3E_3F_3$.
Similarly for P' and Q'.

11. Let AM, BN be parallel and similarly directed radii in the two circles whose centres are A and B, and let MN and AB produced meet at E.

Then $\triangle$s AEM, BEN are mutually equiangular;
$\therefore$ $AE : BE = AM : BN$; *VI.* 4
$\therefore$ E is the external centre of similitude.

Again, let AM, BK be parallel and oppositely directed radii in the two circles whose centres are A and B, and let MK and AB meet at I.

Then $\triangle$s AIM, BIK are mutually equiangular;
$\therefore$ $AI : BI = AM : BK$;
$\therefore$ I is the internal centre of similitude.

12. Let E be the external centre of similitude of the two circles whose centres are A and B; through E let there be drawn the secant $ENQMP$, N and Q being on the circle B, M and P on the circle A: to prove $AM \parallel BN$, and $AP \parallel BQ$.

From E draw EKH a common tangent to the two circles, and join AH, BK.

Then $\triangle$s AEH, BEK are mutually equiangular;

$\therefore \quad AE : BE = AH : BK,$ *VI.* 4

$= AM : BN.$

Now in $\triangle$s AEM, BEN, $\angle AME$ is greater than $\angle AHE$, and $\angle BNE$ is greater than $\angle BKE$;

$\therefore$ $\angle$s AME, BNE are both obtuse;

$\therefore$ $\triangle$s AEM, BEN are similar, *VI.* 7

and $\angle MAE = \angle NBE$;

$\therefore$ AM is $\parallel BN$.

Similarly AP is $\parallel BQ$.

13. It has been proved in the previous deduction that AM is $\parallel BN$, and $AP \parallel BQ$;

$\therefore \quad \angle MAP = \angle NBQ.$ *I.* 34, *Cor.*

Now $\angle MHP$ is supplementary to half $\angle MAP$,

and $\angle NKQ$ is supplementary to half $\angle NBQ$;

$\therefore \quad \angle MHP = \angle NKQ$;

$\therefore$ segment MHP is similar to segment NKQ.

14. Let DI produced (fig. on p. 251 of *Euclid*) meet the inscribed circle at J.

Then IJ, I_1D_1 are parallel and similarly directed radii of the circles DEF, $D_1E_1F_1$;

$\therefore$ D_1J produced passes through A, the external centre of similitude of the circles DEF, $D_1E_1F_1$.

15. The straight line joining the vertex of a triangle to the inscribed point of contact on the base, intersects the escribed radius perpendicular to the base on the escribed circle.

Let D_1I_1 produced (fig. on p. 251 of *Euclid*) meet the escribed circle at J_1.

Then I_1J_1, ID are parallel and similarly directed radii of the circles $D_1E_1F_1$, DEF;

$\therefore$ J_1D produced passes through A, the external centre of similitude of the circles $D_1E_1F_1$, DEF.

16. If H be the middle point of BC, it is also the middle point of DD_1, by (15) of the nineteenth deduction of Book IV.;

$\therefore$ in $\triangle DD_1J$, HI is $\parallel D_1J$; *App. I.* 1

$\therefore$ in $\triangle DD_1A$, HI produced bisects AD. *App. I.* 1, *Cor.* 1

17. The middle point of the base of a triangle, the first escribed centre, and the middle of the line drawn from the vertex to the point of escribed contact on the base are collinear.

For in $\triangle D_1J_1D$, I_1H is $\parallel J_1D$; *App. I.* 1

and in $\triangle D_1J_1A$, I_1H produced bisects AD_1.

App. I. 1, *Cor.* 1

18. Let A and B be the centres of the fixed circles, C the centre of the variable circle which touches them at M, Q: to prove that MQ passes through E or I.

Join AC, BC which pass through N and Q. Let MQ meet the circle B again at N, and join BN.

Then $\angle CMQ = \angle CQM$, *I.* 5

$= \angle BQN$,

$= \angle BNQ$; *I.* 5

$\therefore$ AM is $\parallel BN$;

$\therefore$ MQ passes through E or I, according as AM, BN are similarly or oppositely directed.

19. The potency of E with respect to the variable circle C (fig. to the preceding deduction) is constant.

For $AM : BN = EM : EN$,

$= EM \cdot EQ : EN \cdot EQ$. *VI.* 1

Now $EN \cdot EQ$ is constant, since the circle B is fixed;

$\therefore$ $EM \cdot EQ$, which bears a constant ratio to $EN \cdot EQ$, is constant.

Hence the potency of C with respect to another circle D which touches A and B is constant and $= EM \cdot EQ$;

$\therefore$ E lies on the radical axis of C and D.

20. Let A, B, C be the centres of the three circles, and let their radii be denoted by a, b, c respectively. Let the internal and external centres of similitude of B and C be I_1, E_1; of C and A, I_2, E_2; of A and B, I_3, E_3.

(1) To prove E_1, E_2, E_3 collinear.

Join E_1E_2, E_2E_3, and through B draw $BF \parallel E_2E_3$.

Then $AE_3 : BE_3 = a : b$.

But $AE_3 : BE_3 = AE_2 : FE_2$; *VI.* 2

$\therefore$ $AE_2 : FE_2 = a : b$;

$\therefore$ $AE_2 : a = FE_2 : b$.

Now $AE_2 : a = CE_2 : c$;

$\therefore$ $FE_2 : CE_2 = b : c$,

$= BE_1 : CE_1$;

$\therefore$ BF is $\parallel E_1E_2$. *VI.* 2

But BF was drawn $\parallel E_2E_3$;

$\therefore$ E_1, E_2, E_3 are collinear.

(2) To prove E_2, I_1, I_3 collinear.

Join E_2I_3, and through C draw $CG \parallel E_2I_3$.

Then $AE_2 : CE_2 = a : c$.

But $\quad AE_2 : CE_2 = AI_3 : GI_3;$

$\therefore \quad AI_3 : GI_3 = \ a \ : c;$

$\therefore \quad AI_3 : \ a \ = GI_3 : c.$

Now $\quad AI_3 : \ a \ = BI_3 : b;$

$\therefore \quad GI_3 : BI_3 = \ c \ : b,$

$\qquad\qquad\qquad = CI_1 : BI_1;$

$\therefore$ E_2I_3 passes through I_1;

$\therefore$ E_2, I_1, I_3 are collinear.

Hence also E_1, I_2, I_3 are collinear, as well as E_3, I_1, I_2. [Concise proofs of the collinearity of these sets of three points may be obtained by the method of Transversals.]

LOCI.

1. Let ABC be the given triangle, and let DE be one of the straight lines drawn $\parallel$ BC, and terminated by AB, AC, or AB, AC produced.

   Bisect BC in F; join AF, and let AF or AF produced meet DE at G.

   Then $AG : GD = AF : FB$, *VI.* 4

   $\qquad = AF : FC$,

   $\qquad = AG : GE$; *VI.* 4

   $\therefore$ $GD = GE$, that is, the middle point of DE lies on AF or AF produced.

   Hence AF produced both ways is the locus.

2. Let A be the given point, BC the given straight line, $K : L$ the given ratio.

   Draw any straight line AD to BC; divide AD internally at E and externally at F in the ratio of $K : L$. Through E and F draw parallels to BC. These parallels constitute the locus.

3. Let A be the given point (fig. on p. 104 of *Euclid*), C the centre of the given circle, $K : L$ the given ratio.

   Join AC, and from A draw any straight line AB to the given $\bigcirc^{ce}$.

   Divide AC in D, so that $AD : DC = K : L$; join CB, and draw $DE \parallel CB$.

   Then $AE : EB = AD : DC$, *VI.* 2

   $\qquad = \ K \ : L$;

   that is, the straight line drawn through D, a fixed point

parallel to CB, always cuts AB in the given ratio, however AB may be situated.

Again $\triangle$s ACB, ADE are mutually equiangular;

$\therefore \quad AC : AD = CB : DE.$ VI. 4

But since $AD : DC = K : L$,

$\therefore \quad AC : AD = K + L : K$, by addition;

$\therefore \quad K + L : K = CB : DE$,

that is, DE is a fourth proportional to $K + L$, K, and CB.

Now $K + L$, K, and CB are all constant magnitudes;

$\therefore$ DE must be constant;

$\therefore$ the locus of E is the $\bigcirc^{ce}$ of a circle with D for centre and DE for radius.

If AB' instead of AB be considered as the variable line, and a similar construction be made with regard to it, it may be proved as before that DE', the parallel through D to CB', cuts AB' in the given ratio, and that DE' is a fourth proportional to $K + L$, K, and CB'. Hence the locus of E' is the same as the locus of E.

If $K : L$ be a ratio of greater inequality, divide AC produced at F, so that $AF : FC = K : L$; through F draw $FG \parallel CB$ to meet AB produced at G.

Then it may be proved as before that $AG : GB = K : L$; that FG is a fourth proportional to $K - L$, K, and CB, and that FG is constant.

Hence the locus of G is the $\bigcirc^{ce}$ of a circle with F for centre and FG for radius.

If $K : L$ be a ratio of less inequality, divide CA produced at F so that $AF : FC = K : L$; through F draw $FG \parallel CB$ to meet BA produced at G.

Then as before the locus of G is the $\bigcirc^{ce}$ of a circle whose centre is F and radius FG, a fourth proportional to $L - K$, K and CB.

4. Since the base and the vertical angle of the triangle are given, the locus of the vertex is the arc of a segment of a circle described on the given base, and containing an angle equal to the given vertical angle. Let BC be the given base, A one position of the vertex.

Then H, the middle point of BC, is given, and G the centroid divides HA in the ratio $1 : 2$;

$\therefore$ the locus of G is an arc of a circle, by the previous deduction.

5. Let AB, AC be the two given straight lines, and let the distance of the point from AB be to its distance from AC as $K : L$.

Take any point in AB, and at it draw a perpendicular to AB equal to K; take any point in AC, and at it draw a perpendicular to AC equal to L. Through the ends of these perpendiculars draw parallels to AB, AC respectively, meeting at a point P_1. Join AP_1.

AP_1 produced is part of the locus.

To obtain the complete locus, draw the perpendiculars to AB, AC on both sides of AB, AC; the parallels to AB, AC drawn through the extremities of the perpendiculars will intersect in three other points P_2, P_3, P_4. If P_1 be situated within the angle BAC, P_2 within the angle contained by AC and BA produced, P_3 within the angle contained by BA produced and CA produced, P_4 within the angle contained by AB and CA produced, it will be found that AP_1 and AP_3 are in one straight line, and AP_2 and AP_4 are in one straight line. The complete locus therefore consists of P_1P_3 and P_2P_4 produced indefinitely.

If the two given straight lines be parallel, draw any perpendicular to them, and divide that portion of it intercepted by the two internally and externally in the given ratio. Through the two points of section draw parallels to the given straight lines.

6. If AB and BC subtend equal angles at D,
then $AD : DC = AB : BC$. *VI.* 3
Now AC is known, and $AB : BC$ is known;
$\therefore$ this deduction is the same as the succeeding one.

7. Let BC be the given base, $K : L$ the given ratio.

Divide BC internally at D and externally at E in the ratio $K : L$;
then D and E are fixed points, and DE is a fixed distance.

Let A be any point on the locus of the vertex. Join A to B, D, C, E.

Then $BA : AC = BD : DC$, and $BA : AC = BE : EC$;
$\therefore$ AD bisects $\angle\ BAC$, and AE bisects the supplement of $\angle\ BAC$; *VI.* 3, A
$\therefore$ $\angle\ DAE$ is right;

$\therefore$ the fixed distance DE subtends at any point of the locus a right angle.

$\therefore$ the locus of A is the circle described on DE as diameter.

8. Let ABC be a triangle. From A draw $AX \perp BC$; bisect AX at U, and BC at H.

Then when the inscribed rectangle is infinitely thin as AX, U is the intersection of its diagonals; when it is infinitely thin as BC, H is the intersection of its diagonals;

$\therefore$ the points U and H belong to the locus.

To prove that the locus is UH, let $DEFG$ be a rectangle inscribed in $\triangle\ ABC$, D being on AB, E on AC, and F and G on BC.

Join AH cutting DE at M; through M draw $MN \perp BC$, and cutting UH at O.

Since AH is a median, and DE is $\parallel BC$,

$\therefore$ M is the middle point of DE, and the intersection of the diagonals of $DEFG$ is the middle point of MN.

Since HU is a median of $\triangle\ HAX$, and MN is $\parallel AX$,

$\therefore$ O is the middle point of MN;

$\therefore$ the intersection of the diagonals of $DEFG$ lies on UH.

Corresponding to the rectangles which have two of their vertices situated on CA there will be another straight line KV joining K, the middle of CA, to V, the middle of the perpendicular drawn from B to CA; to the rectangles which have two of their vertices situated on AB there will be a straight line LW joining L, the middle of AB, to W, the middle of the perpendicular drawn from C to AB. Hence the complete locus consists of these three straight lines, which may be proved to be concurrent.

9. Let BD, CE meet at F, and join AF.

Since $\angle\ BAC = \angle\ DAE$, $\therefore$ $\angle\ BAD = \angle\ CAE$;

and $AB : AD = AC : AE$;

$\therefore$ $\triangle$s ABD, ACE are similar, *VI.* 6

and $\angle\ ADF = \angle\ AEF$.

Hence the four points A, D, E, F are concyclic; *III.* 21

$\therefore$ $\angle\ AFC$ is supplementary to $\angle\ ADE$; *III.* 22

$\therefore$ $\angle\ AFC$ is supplementary to $\angle\ ABC$;

$\therefore$ the four points A, B, C, F are concyclic; *III.* 22

$\therefore$ the locus of F is the circle circumscribed about the fixed $\triangle\ ABC$.

10. The rhombus $ABCD$ consists of two equilateral triangles;

$\therefore$ one pair of its angles are each $= \frac{2}{3}$ of 2 rt. $\angle$ s,
and the other pair are each $= \frac{1}{3}$ of 2 rt. $\angle$ s.
Since $\triangle$ s PAD, DCQ are mutually equiangular,

$\therefore$ $PA : AD = DC : CQ$; *VI.* 4

$\therefore$ $PA : AC = AC : CQ$.

Now $\angle\ PAC = \frac{2}{3}$ of 2 rt. $\angle$ s $= \angle\ ACQ$;

$\therefore$ $\triangle$ s PAC, ACQ are similar, *VI.* 6

and $\angle\ ACP = \angle\ CQA$;

$\therefore$ $\angle\ CAM + \angle\ ACM = \angle\ CAQ + \angle\ CQA$,
$= \frac{1}{3}$ of 2 rt. $\angle$ s;

$\therefore$ $\angle\ AMC = \frac{2}{3}$ of 2 rt. $\angle$ s;

$\therefore$ the four points A, B, C, M are concyclic; *III.* 22

$\therefore$ the locus of M is the $\bigcirc^{ce}$ of the circle circumscribed about $\triangle\ ABC$.

11. Let A be the vertex of one of the series of triangles on the fixed base BC; then the locus of A is the $\bigcirc^{ce}$ of a circle, since BA is a given length.

Let the bisector of $\angle\ ABC$ meet AC at D;
then $CD : DA = CB : BA$, *VI.* 3
$=$ a given ratio;

$\therefore$ by the third deduction of this set, the locus of D is the $\bigcirc^{ce}$ of a circle.

If the bisector of the exterior angle at B, meets CA produced at E, then $CE : EA = CB : BA$, *VI.* A
$=$ a given ratio;

$\therefore$ by the third deduction of this set, the locus of D is the $\bigcirc^{ce}$ of a circle.

The centre of the first circle is on CB, the centre of the second on CB produced.

12. Let the tangents from C and D to the circle intersect the tangent at A in the points K, L respectively, and each other in M.

From M draw $MP \perp BA$ produced.

Join O, the centre of the circle, with K, C, and E the point of contact of KC.

Then $\angle\ OKC = \frac{1}{2}\ \angle\ AKC$, $\angle\ OCK = \frac{1}{2}\ \angle\ BCK$.

But $\angle\ AKC + \angle\ BCK = 2$ rt. $\angle$ s;

$\therefore$ $\angle\ OKC + \angle\ OCK =$ a right angle;

$\therefore$ $\angle\ KOC$ is right;

$\therefore$ $EK \cdot EC = EO^2$; *VI.* 8, 17

$\therefore \quad AK \cdot BC = EO^2$, by the first deduction from III. 17.
Now $BD \cdot BC = AB^2 = 4\,EO^2$; *VI.* 8, 17
$\therefore \quad BD = 4\,AK.$
Similarly $BC = 4\,AL$; $\therefore\ CD = 4\,KL$;
$\therefore \quad MC = 4\,MK$; $\therefore\ PB = 4\,PA.$
Hence P is a fixed point, and the locus of M is a straight line parallel to CD, and distant from it $\frac{3}{5}$ of the diameter.

13. Let BE, CD meet at M, and BD, CE at N.

Through D draw $DF \parallel BE$, and through E draw $EG \parallel BD$, and let these parallels meet the straight line ABC in F and G.

Then $AF : FB = AD : DE$, *VI.* 2
$= $ a fixed ratio;
$\therefore\ BF$ is fixed, since AB is fixed;
$\therefore \quad CB : BF$ is a fixed ratio.
Now $\quad CM : MD = CB : BF$; *VI.* 2
$\therefore$ the locus of M is a straight line parallel to XY, by the second deduction of this set.

Again $AG : GB = AE : ED$, *VI.* 2
$= $ a fixed ratio;
$\therefore\ BG$ is fixed, since AB is fixed;
$\therefore\ CB : BG$ is a fixed ratio.
Now $\quad CN : NE = CB : BG$; *VI.* 2
$\therefore$ the locus of N is a straight line parallel to $X'Y'$, by the second deduction of this set.

14. Let AB be a straight line passing through O, A being on XY, and B on $X'Y'$, and let ABC be an equilateral triangle on AB.

From C draw $CG \perp XY$; from O draw $DOE \perp XY$ and $X'Y'$, D being on XY and E on $X'Y'$;
from O draw $OF \perp CG$, and join OC.

Then $\triangle$s ADO, CFO are mutually equiangular, since the sides of the one are perpendicular to the sides of the other;
$\therefore\ OA : OC = OD : OF.$ *VI.* 4
But $OA : OC$ is a constant ratio, being $= 1 : \sqrt{3}$;
$\therefore\ OD : OF$ is a constant ratio;
$\therefore\ OF$ is constant, since OD is constant;
$\therefore$ the locus of C is a straight line drawn $\perp XY$.
The complete locus will consist of two straight lines $\perp XY$, on opposite sides of DE, and equidistant from it.

Since $OD : OF = 1 : \sqrt{3}$,

F is the vertex of an equilateral triangle on DE;

$\therefore$ the locus passes through the vertices of the two equilateral triangles described on DE.

15. Let OAB cut XY and $X'Y'$ at A and B, and on AB let one of the similar triangles ABC be described: to find the locus of C.

From O draw $OFG \perp XY$ and $X'Y'$, F being on XY and G on $X'Y'$, and on FG describe $\triangle FGH$ similar to $\triangle ABC$. Join OH, OC, HC.

Then $OF : OA = FG : AB$,

$= FH : AC$.

Now $\angle OFH = \angle OAC$, since they are supplements of equal angles;

$\therefore$ $\triangle$s OFH, OAC are similar, *VI.* 6

and $OF : OA = OH : OC$.

But since $\angle FOA = \angle HOC$,

$\therefore$ $\triangle$s OFA, OHC are similar; *VI.* 6

$\therefore$ $\angle OHC = \angle OFA$ = a right angle;

$\therefore$ the locus of C is the perpendicular to OH, drawn through the point H.

The complete locus will consist of two straight lines symmetrically situated with respect to OG.

[The last eight examples are given in Vuibert's *Journal de Mathématiques Élémentaires.*]

MISCELLANEOUS.

1. Let ABC be a triangle, AH the median from A, G the centroid, S the circumscribed centre.

Join SH, SG, and let AX, the perpendicular from A on BC, meet SG produced at O.

Then $\triangle$s AGO, HGS are mutually equiangular;

$\therefore$ $OG : SG = AG : HG$, *VI.* 4

$= 2 : 1$,

that is, AX cuts SG produced, so that $OG = 2\,SG$.

Hence also BY, CZ, the perpendiculars from B, C on CA, AB, cut SG produced, so that $OG = 2\,SG$;

that is, the perpendiculars AX, BY, CZ are concurrent, S, G, O are collinear, and $SG : GO = 1 : 2$.

2. The centre M of the medioscribed circle is the middle point of OS. *App. IV. 2, Cor. 2*

Let the distance OS be denoted by 6;
then by the previous deduction $OG = 4$, $SG = 2$.
But $OM = 3$; $\therefore MG = 1$.
Hence $OM : MG = OS : SG$,
and O, M, G, S form a harmonic range.

The internal and external centres of similitude of the circumscribed and medioscribed circles are found by dividing SM internally and externally in the ratio of the radii, that is, in the ratio 2 : 1.
Now $SG : GM = 2 : 1$, and $SO : OM = 2 : 1$;
$\therefore G$ and O are the two centres of similitude.

3. Let P be any point on the $\bigcirc^{ce}$ of the circumscribed circle; join OP, and let it meet the $\bigcirc^{ce}$ of the medioscribed circle at P'.

Join SP, MP'.

Then SP is $\parallel MP'$, by the twelfth example of Centres of Similitude;
and $OP : OP' = SP : MP'$,
$= 2 : 1.$

4. Let P be any point on the $\bigcirc^{ce}$ of the circumscribed circle; join GP, and let it be produced in the direction PG to meet the $\bigcirc^{ce}$ of the medioscribed circle at P'.

Join SP, MP'.

Then SP is $\parallel MP'$, and $GP : GP' = SP : MP'$,
$= 2 : 1.$

5. Let HKL be the centroidal or median triangle of $\triangle ABC$;
then $\triangle$s ABC, HKL are similar and oppositely situated.
Now $\triangle ABC$ is the median triangle of $\triangle A'B'C'$ (fig. to nineteenth deduction on p. 357); and I' is the inscribed centre of $\triangle A'B'C'$;
$\therefore I$ bears the same relation to $\triangle HKL$ that I' does to $\triangle ABC$;
that is, in $\triangle$s ABC, HKL, I' and I are homologous points.
But in these same triangles A and H are homologous points;
$\therefore I'I$ and AH intersect at the centre of similitude of $\triangle$s ABC, HKL.
Let G be the point of intersection of $I'I$ and AH;
then $I'G : IG = AG : HG = AB : HK = 2 : 1$.
Hence G is the centroid.

6. Since G is the centre of similitude of $\triangle$s ABC, HKL, which are oppositely situated, it is therefore the internal centre of similitude of the circles inscribed in ABC, HKL. Now I is the inscribed centre of $\triangle$ ABC;

$\therefore$ the inscribed centre of $\triangle$ HKL must be on IG produced. Take $GJ = \frac{1}{2}\, GI$; then J is the point required.

For the radii of the circles inscribed in $\triangle$s ABC, HKL are in the ratio $2 : 1$, and $GI : GJ = 2 : 1$.

To prove that J is the middle point of $I'I$, denote the distance $I'I$ by 6; then $I'G = 4$, and $IG = 2$, by the previous deduction;

$\therefore$ $GJ = 1$, and $IJ = 3 = \frac{1}{2}\, II'$.

7. The points A, I' in $\triangle$ ABC correspond to the points H, I in $\triangle$ HKL;

$\therefore$ AI' is $\parallel$ HI, and $= 2\, HI$.

Let HJ meet AI' in U;

then $\triangle$s IJH, $I'JU$ are congruent, and $IH = I'U$; *I.* 26

$\therefore$ $AI' = 2\, I'U$.

8. Since $IG : GJ = 2 : 1$, and $II' : I'J = 6 : 3$,

$\therefore$ $IG : GJ = II' : I'J$;

$\therefore$ I', J, G, I form a harmonic range.

Because IJ, the distance between the centres of the circles inscribed in $\triangle$s ABC, HKL, is divided internally in G and externally in I' in the ratio of the radii, $2 : 1$,

$\therefore$ G and I' are the internal and external centres of similitude.

9. Let U, V, W be the middle points of AI', BI', CI', and join UV, VW, WU.

Then UV is $\parallel$ AB, and $= \frac{1}{2}\, AB$,
VW is $\parallel$ BC, and $= \frac{1}{2}\, BC$,
WU is $\parallel$ CA, and $= \frac{1}{2}\, CA$; *App. I.* 1

$\therefore$ $\triangle$s ABC, UVW are similar.

And they are similarly situated, I' being their centre of similitude.

Now since I' is the centre of similitude of $\triangle$s ABC, UVW, which are similarly situated, it is the external centre of similitude of the circles inscribed in $\triangle$s ABC, UVW.

But I is the inscribed centre of $\triangle$ ABC;

$\therefore$ the inscribed centre of $\triangle$ UVW must lie on II'.

Bisect II' at J; then $II' : I'J = 2 : 1$;

$\therefore$ J is the inscribed centre of $\triangle$ UVW;

$\therefore$ $\triangle$s HKL, UVW have the same inscribed centre.

And they have the same inscribed radius, for they are congruent;

∴ they have the same inscribed circle.

10. The properties are:

I_1, G, I_1' are collinear, and $I_1'G = 2\,I_1G$.

The middle point J_1 of I_1I_1' is the centre of the first escribed circle of the median $\triangle\ HKL$.

The points H, J_1, and the middle point of AI_1' are collinear.

The points I_1', J_1, G, I_1 form a harmonic range, and G and I_1' are the internal and external centres of similitude of the first escribed circles of $\triangle$s ABC, HKL.

The first escribed circle of $\triangle\ HKL$ is also the first escribed circle of the triangle formed by joining the middle points of AI_1', BI_1', CI_1'.

The corresponding properties for the points I_2', I_3' are obtained from the preceding by changing the subscripts, and the word 'first.'

[See *Proceedings of The Edinburgh Mathematical Society*, Vol. II., Session 1883–4, pp. 2–4.]

11. Suppose O to be the centre of the circle ABC, and OB, OE to be joined: it is required to prove $\triangle\ OBE$ equilateral, from which the conclusion follows at once.

Join OA, OP, AP.

Then	$\angle\ OPA = \angle\ OPB;$	*I.* 8
∴	$\angle\ APB = 2\,\angle\ OPB,$	
	$= 2\,\angle\ OBP.$	*I.* 5
Now	$\angle\ APB = \angle\ AEB;$	*III.* 21
and	$\angle\ APB = 2\,\angle\ ADB,$	*III.* 20
and	$\angle\ AEB = \angle\ EDB + \angle\ EBD;$	*I.* 32
∴	$2\,\angle\ ADB = \angle\ EDB + \angle\ EBD;$	
∴	$\angle\ EDB = \angle\ EBD;$	
∴	$\angle\ AEB = 2\,\angle\ EBD;$	
∴	$\angle\ APB = 2\,\angle\ EBD;$	
∴	$\angle\ OBP = \angle\ EBD;$	
∴	$\angle\ OBP + \angle\ PBE = \angle\ EBD + \angle\ PBE;$	
∴	$\angle\ OBE = \angle\ PBD.$	

But $\angle\ PBD$ is an angle of an equilateral triangle;

∴ $\angle\ OBE$ is an angle of an equilateral triangle.

And since $OB = OE$, $\triangle\ OBE$ is equilateral.

[The preceding proof is due to T. S. Davies, and is given, somewhat confusedly, on p. 542 of the volume cited.]

12. With B as centre, and BC as radius, describe a circle cutting AB at E and AB produced at F; and join FC.

Then FC is $\parallel BD$; *III.* 20, *I.* 28

$\therefore \quad FB : CD = FA : CA$; *VI.* 2

$\therefore \quad FB^2 : CD^2 = FA^2 : CA^2$, *VI.* 22

$= FA^2 : FA \cdot AE$, *III.* 36

$= FA : AE$; *VI.* 1

$\therefore \quad BC^2 : CD^2 = FA : AE$;

$\therefore BC^2 : BC^2 - CD^2 = FA : FE$, by subtraction.

Double the first antecedent, and halve the last consequent; then $2\,BC^2 : BC^2 - CD^2 = FA : FB$,

$= CA : CD$. *VI.* 2

[This solution is attributed to James Golius (1596–1667), Professor of Arabic in Leyden.]

13. In the following proof implicit reference will be made to the properties established in Appendix I. 3, 4, 5, the twentieth, and (1), (4), of the twenty-fourth, deductions of Book IV. Also, as in the twenty-ninth and thirtieth deductions of Book IV., $H'U$, $K'V$, $L'W$ are the diameters of the circle ABC respectively $\perp$ BC, CA, AB; G_0, G_1, G_2, G_3 are the centroids of $\triangle$s $I_1I_2I_3$, II_2I_3, II_3I_1, II_1I_2.

Since H', K', L' are the middle points of I_2I_3, I_3I_1, I_1I_2,

$\therefore$ S_1, the circumscribed centre of $\triangle$ $I_1I_2I_3$, is the orthocentre of $\triangle$ $H'K'L'$.

Since I is the orthocentre of $\triangle$ $I_1I_2I_3$,

$\therefore$ G_0 is situated on IS_1, so that $IG_0 : S_1G_0 = 2 : 1$;

and G_1 is situated on IH', so that $IG_1 : H'G_1 = 2 : 1$;

$\therefore IG_0 : S_1G_0 = IG_1 : H'G_1$;

$\therefore G_0G_1$ is $\parallel S_1H'$. *VI.* 2

Similarly G_0G_2 is $\parallel S_1K'$, and G_0G_3 is $\parallel S_1L'$.

Hence $\triangle$ $G_1G_2G_3$ is similar and similarly situated to $\triangle$ $H'K'L'$; and as S_1 is the orthocentre of $\triangle$ $H'K'L'$, G_0 is the orthocentre of $\triangle$ $G_1G_2G_3$;

$\therefore$ G_1, G_2, G_3 are the orthocentres of $\triangle$s $G_0G_2G_3$, $G_0G_3G_1$, $G_0G_1G_2$.

14. Let $ABCDEF$ be a complete quadrilateral, E being the point where AB and DC meet, and F the point where BC and AD meet, and AC, BD, EF its three diagonals.

The three sides AB, BC, CD of the quadrilateral form $\triangle$ BCE. Draw the three perpendiculars BX, CY, EZ, and let P_1 be the orthocentre of $\triangle$ BCE.

The circle on BD as diameter passes through X;
the circle on AC as diameter passes through Y; and
the circle on EF as diameter passes through Z.
Now the potencies of P_1 with respect to these three circles are $P_1B \cdot P_1X$, $P_1C \cdot P_1Y$, $P_1E \cdot P_1Z$.
But $P_1C \cdot P_1Y = P_1B \cdot P_1X$, since C, X, B, Y are concyclic;
$= P_1E \cdot P_1Z$, since B, X, E, Z are concyclic;
$\therefore$ P_1 has equal potencies with respect to these three circles.

The sides BC, CD, DA of the quadrilateral form $\triangle\ CDF$, and if P_2 denote its orthocentre, it will be found that P_2 has equal potencies with respect to the three circles.
Similarly P_3, the orthocentre of $\triangle\ AED$ formed by the sides CD, DA, AB of the quadrilateral, and P_4 the orthocentre of $\triangle\ ABF$ formed by the sides DA, AB, BC of the quadrilateral, have equal potencies with respect to the three circles.
Hence (App. III. 2, Cor. 6) these three circles have the same radical axis, on which P_1, P_2, P_3, P_4 are situated. Hence also the centres of these three circles, which are the middle points of the diagonals, are in the same straight line, and that straight line is perpendicular to the radical axis.
[This solution is due to T. T. Wilkinson. See *The Lady's and Gentleman's Diary* for 1855, p. 58.]

15. (*a*) Let LMN be the circle, AB, CD, EF the three given straight lines: to inscribe a $\triangle\ LMN$ so that LM may be $\parallel AB$, $MN \parallel CD$, and $NL \parallel EF$.

Since AB and CD are fixed straight lines, the angle contained by them, if they be produced to intersect, is fixed;
$\therefore$ $\angle\ LMN$ is fixed in magnitude;
$\therefore$ the chord LN is fixed in magnitude.
The problem therefore is reduced to placing in the circle a chord LN equal to the fixed magnitude, and parallel to EF, which is the seventh deduction from IV. 1.

The fixed magnitude to which LN is equal may be found by taking any point G on the ⊙ce, drawing the chord $GH \parallel AB$, and the chord $HK \parallel CD$; then $GK = LN$.
(*b*) Let AB, CD be the two given straight lines, E the given point: to inscribe a $\triangle\ LMN$ so that LM may be $\parallel AB$, $MN \parallel CD$, and that LN may pass through E.

Find GK as before, and through E draw, by the fifth or sixth deduction from IV. 1, the chord $LN = GK$.

(*c*) Let A and B be the given points, CD the given straight line:
to inscribe a $\triangle\ LMN$ so that LM may pass through A, LN through B, and that MN may be $\parallel CD$.

Suppose $\triangle\ LMN$ inscribed in the circle as is required. Join AB, through N draw the chord $NE \parallel AB$, join EM, and let AB and EM, produced if necessary, meet at F.

Since EN is $\parallel AB$, and MN is $\parallel CD$,

$\therefore\ \angle\ ENB =$ the angle between AB and CD,

$=$ a fixed magnitude;

$\therefore$ the chord ME is fixed in magnitude.

To fix the position of ME, and consequently that of MN, it will be necessary to find the position of F.

Now $\angle\ AFM = \angle\ MEN$, *I.* 29

$= \angle\ MLN$; *III.* 21

$\therefore$ the four points F, B, L, M are concyclic; *III.* 22, *Cor.*

$\therefore\quad AF \cdot AB = AM \cdot AL.$ *III.* 35, or *Cor.*

But AB is known, and $AM \cdot AL$ is fixed, since it is the potency of A with respect to the circle;

$\therefore\ AF$ can be found.

(*d*) Let A, B, C be the three given points:
to inscribe a $\triangle\ LMN$ so that LM may pass through A, LN through B, and MN through C.

Suppose $\triangle\ LMN$ inscribed in the circle as is required. Join AB, through N draw the chord $NE \parallel AB$, join EM, and let AB and EM, produced if necessary, meet at F.

Then, as in the preceding case, the position of F may be determined, and a $\triangle\ EMN$ inscribed in the circle having EM passing through F, MN through C, and $EN \parallel AB$. This determines the position of MN, and consequently that of $\triangle\ LMN$.

Since there are in general two solutions to the fifth, sixth, and seventh deductions from IV. 1, there will be in general two $\triangle$s LMN satisfying the required conditions in each of the four cases. These triangles will be symmetrically situated with respect to the centre of the circle.

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